Two-time Green's functions and spectral density method in nonextensive quantum statistical mechanics

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We extend the formalism of the thermodynamic two-time Green's functions to nonextensive quantum statistical mechanics. Working in the optimal Lagrangian multiplier representation, the q-spectral properties and the methods for a direct calculation of the two-time q Green's functions and the related q-spectral density (q measures the nonextensivity degree) for two generic operators are presented in strict analogy with the extensive ($q=1$) counterpart. Some emphasis is devoted to the nonextensive version of the less known spectral density method whose effectiveness in exploring equilibrium and transport properties of a wide variety of systems has been well established in conventional classical and quantum many-body physics. To check how both the equations of motion and the spectral density methods work to study the q-induced nonextensivity effects in nontrivial many-body problems, we focus on the equilibrium properties of a second-quantized model for a high-density Bose gas with strong attraction between particles for which exact results exist in extensive conditions. Remarkably, the contributions to several thermodynamic quantities of the q-induced nonextensivity close to the extensive regime are explicitly calculated in the low-temperature regime by overcoming the calculation of the q grand-partition function.

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I. INTRODUCTION

The method of thermodynamic Green's functions (GFs) [1–10] is a powerful tool in ordinary statistical mechanics for exploring the equilibrium and transport properties of a large variety of many-body systems. These functions are related to important physical quantities, and hence their calculation constitutes one of the basic problems of extensive thermostatistics.

Remarkably, the extraordinary effectiveness of the GF technique in quantum many-body physics [1–7] has stimulated a lot of research activity to extend this successful method also to study extensive classical statistical mechanics [8–10]. In the context of the two-time GFs [2,3] the foundations of the classical formalism were introduced by Bogoliubov and Sadovnikov [8] four decades ago and further systematic developments in this direction were performed only many years later [2,9]. Additionally, the classical counterpart of the quantum Matsubara GF framework [5] was achieved in Ref. [10]. So also the two-time GF formalism in extensive classical statistical mechanics can be now considered well established. Although not currently used in the literature, it has been successfully employed to study the thermodynamics and the transport properties of several classical many-body systems [2,8–10] also involving phase transitions and critical phenomena [9].

Various (also numerical) methods have been developed for the calculation of the two-time GFs both in quantum and classical statistical physics [1–7]. In addition to the most known "equations-of-motion method" (EMM), the related "spectral density method" (SDM), originally formulated by Kalashnikov and Fradkin [7] within the quantum statistical mechanics context, appears to be a very promising nonperturbative approach to perform reliable studies of the macroscopic properties of classical and quantum many-particle systems [7,9], avoiding an explicit calculation of the partition function.

At the present time, the situation does not appear so well established in nonextensive statistical mechanics which has attracted an increasing interest since the seminal proposal by Tsallis made almost 20 years ago [11,12]. This framework can be regarded as a generalization of the Boltzmann-Gibbs statistical mechanics to properly describe macroscopic memory effects (e.g., non-Markovian stochastic processes) and, generally speaking, systems exhibiting nonergodic microscopic dynamics [13–15].

Despite its elegant formalism, the Tsallis thermostatistics is affected by some intrinsic difficulties in performing explicit analytical calculations for realistic many-body systems. Nevertheless, by resorting to different techniques and approximations as generalizations of the extensive ones, the Tsallis theory has been successfully applied to a wide variety of systems for which nonextensivity effects are not negligible [13–15] and must be taken into account for a proper comparison with experiments. Known theoretical tools have been employed and adapted in the Tsallis framework such as, for instance, linear response theory [16], perturbation and variational methods [17], path integral [18,19], Monte Carlo [20–24], and molecular dynamics [21] techniques, and many others. Less attention, however, has been devoted to the extension, to the nonextensive many-body world, of the well-

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established thermodynamic GF technique [1–10] in ordinary quantum and classical statistical mechanics.

Some years ago the GF method in the Kadanoff-Baym framework [4] was generalized to the Tsallis quantum statistical mechanics adopting the second-quantized representation for many-particle systems [25,26]. In these works, the q GFs for a nonextensive many-body system were formally expressed in terms of parametric integrals over the corresponding extensive ($q=1$) quantities, q denoting the so-called Tsallis parameter which measures the nonextensivity degree. Of course, this method may be really useful when the many-body problem for the extensive counterpart has been solved, but unfortunately, this is possible only in a limited number of situations. In any case, the crucial step to calculate nontrivial contour integrals in the complex space constitutes an additional formidable problem which would require, generally, further approximations. Thus, many-body methods which allow one to perform direct calculations of the q properties, overcoming the *a priori* knowledge of the related extensive ones, are desirable. Motivated by the conviction that a direct q GF method may provide new and effective calculation techniques to deal with nontrivial nonextensivity problems, we have recently extended [27,28] the Bogoliubov-Sadovnikov two-time GF framework [8,9] and the related SDM [9] to nonextensive classical statistical mechanics working conveniently within the so-called optimal Lagrangian multiplier (OLM) representation suggested in Ref. [29]. This choice avoids some intrinsic difficulties involved in other ones and allows simplified analytical and numerical calculations. In any case, our suggestion is quite general and can be extended to different contexts preserving the physical content, consistently with the equivalence of the current four versions of the Tsallis statistics [30].

The aim of the present article is to extend the two-time GF formalism to the quantum nonextensive statistical mechanics within the OLM representation. Particular emphasis will be devoted to the spectral density (SD) and its spectral decomposition due to their relevance for practical calculations.

The q EMM and q SDM are here presented as powerful tools for a direct calculation of the two-time q GFs and q SD, respectively. Besides, in parallel with the extensive counterpart [9], we outline the key ideas to explore the dispersion relation and the damping of elementary and collective excitations in nonextensive many-body systems. Finally, with the aim to show how the two-time GF method works when the Tsallis q distribution is involved, we present analytical calculations for a nontrivial high-density Bose gas with strong attraction between the particles for which exact results exist in the extensive case [31,32]. The effects of the q-induced nonextensivity will be consistently explored in the low-temperature regime using both the q EMM and q SDM.

The article is organized as follows. In Sec. II, we present a summary of the basic ingredients of the nonextensive quantum statistical mechanics which will be useful for the next developments. In particular, we focus on the quantum OLM representation which appears more convenient for our purposes. Section III is devoted to the formulation of the two-time GF method in the Tsallis quantum statistics, with emphasis on the spectral properties. In Sec. VI, we present a nonextensive version of the EMM and SDM. Here, the q SDM is properly implemented to offer the possibility to study systematically the dispersion relation and the damping of excitations within a unified formalism. Section V deals with the application of these methods to a model which describes a high density Bose gas with strong attraction between the particles. Concluding remarks are drawn in Sec. VI. Two appendixes close the article. In Appendix A the recently proposed [27,28] two-time GF method in Tsallis classical statistical mechanics is shortly reviewed with the aim to point out the substantial differences between the nonextensive quantum and classical frameworks. Appendix B summarizes some mathematical details.

II. BASIC INGREDIENTS OF NONEXTENSIVE QUANTUM STATISTICAL MECHANICS

The Tsallis nonextensive thermostatistics constitutes today a new paradigm in the field of statistical mechanics. The key problem of the Tsallis framework is to find the appropriate von Neumann density operator ρ which maximizes the generalized q entropy (in units $k_B=\hbar=1$)

$$S_q = \frac{1 - T_r(\rho^q)}{q-1}, \tag{1}$$

subject to appropriate constraints related to the evaluation of the mean or expectation value of the observables within the nonextensive scenario. Here $T_r(\cdots)$ stands for the usual trace operator.

In the literature, four possible choices have been considered for the evaluation of q expectation values $\langle\cdots\rangle_q$.

(i) The original Tsallis proposal [11]

$$\langle A\rangle_q = T_r(\rho A), \tag{2}$$

where the Hermitian operator A corresponds to a generic observable $\mathcal{A}$. For a system with Hamiltonian H, this implies the canonical representation

$$\rho = [(1-q)(\alpha+\beta H)/q]^{1/q-1}, \tag{3}$$

in terms of the two Lagrange multipliers α and β. Unfortunately, this choice involves some troubles related to the Lagrange multiplier α.

(ii) The Curado-Tsallis (CT) choice [33]

$$\langle A\rangle_q = T_r(\rho^q A), \tag{4}$$

which yields the canonical result

$$\rho = Z_q^{-1}[1-(1-q)\beta H]^{1/1-q}, \tag{5}$$

with

$$Z_q = T_r[1-(1-q)\beta H]^{1/1-q}. \tag{6}$$

This avoids the explicit presence of the multiplier α, but has the disadvantage to exhibit unnormalized mean values ($\langle 1\rangle_q \neq 1$).

(iii) The Tsallis-Mendes-Plastino option [12]

$$\langle A\rangle_q = \frac{T_r(\rho^q A)}{T_r(\rho^q)}, \tag{7}$$

implying the canonical solution

$$\rho = Z_q^{-1}\left[1 - \frac{(1-q)\beta}{T_r(\rho^q)}(H - U_q)\right]^{1/1-q}, \tag{8}$$

with

$$Z_q = T_r\left[1 - \frac{(1-q)\beta}{T_r(\rho^q)}(H - U_q)\right]^{1/1-q}, \tag{9}$$

where $U_q = \langle H \rangle_q$ is the q internal energy. Here we have $\langle 1 \rangle_q = 1$ normalized q mean values, but troubles occur again in obtaining ρ due to the presence of $T_r(\rho^q) \equiv (Z_q)^{1-q}$ in Eqs. (8) and (9) (self-referential problem).

(iv) The OLM improvement [29] which preserves the q entropy (1), but replaces the Tsallis-Mendes-Plastino-like constraints by "centered" mean values. Essentially, the general OLM procedure consists in maximizing the Tsallis generalized entropy (1) subject to the constraints

$$T_r \rho = 1 \tag{10}$$

and

$$T_r[\rho^q (A_j - \langle A_j \rangle_q)] = 0 \quad (j = 1, \ldots, M), \tag{11}$$

where the generalized mean values $\langle A_j \rangle_q = T_r(\rho^q A_j)/T_r(\rho^q)$ for M relevant A_j $(j = 1, \ldots, M)$ observables are assumed to be known *a priori* and hence are regarded as constraints in the variational approach to the nonextensive thermostatistics.

In the canonical representation this yields

$$\rho = Z_q^{-1}[1 - (1-q)\beta(H - U_q)]^{1/1-q}, \tag{12}$$

where the generalized partition function Z_q is now given by

$$Z_q = T_r[1 - (1-q)\beta(H - U_q)]^{1/1-q}. \tag{13}$$

As we see, the OLM framework avoids all the inconveniences occurring in the previous choices. Besides, in Eqs. (12) and (13), the Lagrange multiplier β does not depend on the partition function and it is identified as the inverse physical temperature [29,30] $(\beta = 1/T)$, consistently with the zeroth law of thermodynamics.

The extension of the canonical OLM prescription to the grand-canonical ensemble can be simply obtained from Eqs. (12) and (13) by replacing H and U_q with $\mathcal{H} = H - \mu N$ and $\mathcal{U}_q = \langle \mathcal{H} \rangle_q = U_q - \mu \langle N \rangle_q$, where μ is the chemical potential and N denotes the operator describing the total number of particles in a system. Other ensemble representations can be obtained similarly.

It must be emphasized that the four versions (i)–(iv) of the Tsallis thermostatistics are equivalent in the sense that the probability distribution for each of them can be easily derived from any other of them by using appropriate transformation rules [30]. In any case, for practical calculations one must select the most convenient version case by case.

As already mentioned, in the present work we find it convenient to adopt the OLM version to formulate the two-time GF method in nonextensive statistical mechanics, but the framework can be easily extended to different versions by using the prescriptions given in Ref. [30].

To close this short review of the basic elements of the Tsallis thermostatistics, we mention two relevant features of the nonextensive scenario.

The first one, tacitly assumed before, is that the basic OLM canonical operator $\hat{f}_q = 1 - (1-q)\beta(H - U_q)$ must be positive definite. This means that, for any specific nonextensive problem, one must take into account only microstates $|n\rangle$, with $H|n\rangle = E_n|n\rangle$, satisfying the cutoff condition [12]

$$E_n < \frac{1}{(1-q)\beta} + U_q, \quad \text{if } q < 1, \tag{14a}$$

$$E_n > \frac{1}{(1-q)\beta} + U_q, \quad \text{if } q > 1, \tag{14b}$$

where the quantity $1/(1-q)\beta + U_q$ could have, in principle, positive or negative sign.

The second crucial question concerns the physical origin of the nonextensivity parameter q. Unfortunately, little has been definitively proposed in terms of first principles to explain the appearance of the Tsallis thermostatistics in many real situations and to understand the physical meaning of the elusive parameter q. We cite below some general ideas which, in our opinion, may have a seminal role in future research activity on the subject.

After several attempts [15], a well-defined scenario emerged almost seven years ago, now called the "finite-heat-bath picture." First Almeida [34] derived the Tsallis power-law probability distribution from first principles assuming exactly constant the heat capacity C_B of the thermal bath in contact with the system of interest. Here, the physical meaning of q is simply expressed in terms of C_B. Specifically, when it is finite, $q \neq 1$ and one has the Tsallis distribution, while for an infinite heat capacity of the bath, $q = 1$ and one recovers the conventional exponential distribution. This picture has been further elaborated [35,36], but a substantial progress was performed in Ref. [36] where, via a model-free derivation of the Tsallis statistics (i.e., without resort to the microscopic details of a system and of its surrounding), it is shown that the finiteness of the heat capacity of the environment is not necessarily due to only the finiteness of its degrees of freedom, as reductively assumed in previous treatments.

A more general point of view was reported in Ref. [37] where the parameter q was interpreted as a measure of fluctuations of the parameters (as the temperature) in the exponential distribution and expressed in terms of the variance of their inverse. The superstatistics [38], subsequently developed as a generalization of this idea, consists in a superposition of different statistics relevant for driven nonequilibrium systems with complex dynamics in stationary states with large fluctuations of intensive parameters (e.g., the temperature, chemical potential, energy dissipation, etc.) on long time scales. It explains the emergence of the power-law statistics for real systems as a result of fluctuations in their environments, supporting the Tsallis framework as a special case. In this "fluctuational picture," the nonextensivity parameter q is found to be directly related to proper stochastic processes or constraints imposed to the systems and, in the absence of fluctuations, the extensive case is consistently reproduced. Remarkably, the superstatistics seems to offer a general formalism for treating nonequilibrium stationary

states of complex systems which exhibit dynamics that can be decomposed into several dynamics on different time scales. Some effort has been performed recently [39] to legitimate this theory as a true statistical mechanics framework, but several aspects remain to be further elaborated and clarified.

Other, rather fragmentary, proposals to justify the Tsallis statistics exist, but we mention here only some of them that, in our opinion, may be relevant in nonextensive many-body physics.

There is a broad consensus of opinion that the nonextensivity may occur also in cases where long-range interactions play a relevant role, the gravitational and Coulomb forces being important examples with laboratory and astrophysics implications [15]. However, no definitive statistical foundation for long-range interacting systems, such as the self-gravitating ones, has still been well established and, in particular, the true physical nature of q in terms of the range of the forces and the space dimensions has not been well understood yet [40].

Also confining traps for interacting bosonic or fermionic systems may induce nonextensivity effects [41] due to the interplay between the particle interactions and the trapping potential. Moreover, interesting findings about the physical origin and the measurement of the parameter q have been obtained recently for complex magnetic systems such as the manganite $La_{0.7}Sr_{0.3}MnO_3/MgO$ and the amorphous alloy $Cu_{90}Co_{10}$ [42]. Such compounds seem to embody three basic ingredients which may induce nonextensivity: long-range interactions, clusters with fractal shapes, and intrinsic inhomogeneity. With these features in mind it has been shown [42], via theory (at a mean-field approximation level) and scanning tunneling spectroscopy measurements, that the magnetic properties of these materials can be properly described using the Tsallis thermostatistics, q measuring the competition of the intrinsic inhomogeneity and dynamics.

Finally, for our demonstrative application to a high-density Bose gas performed in a next section, we have in mind some of the nonextensivity mechanisms outlined before. For instance, we think about a system of bosons in a fluctuating environment, in confining traps, or confined to a self-gravitational field as in nonrelativistic boson stars [43].

We conclude this section noting that it is frequently convenient to use the so-called q exponential function

$$e_q^x \equiv [1 + (1-q)x]^{1/1-q}, \tag{15}$$

which simplifies sensibly the formalism of the nonextensive statistical mechanics. One can verify that this function (a) yields $e_1^x = e^x$ for $q \to 1$, (b) for $q > 1$ vanishes as a power law when $x \to -\infty$ and diverges at $x = -1/(1-q)$, and (c) for $q < 1$ has a cutoff at $x = -1/(1-q)$ below which it becomes identically zero.

Using the function (15), the basic expressions (12) and (13) for the OLM canonical formulation can be rewritten as

$$\rho = Z_q^{-1} e_q^{-\beta(H-U_q)} \tag{16}$$

and

$$Z_q = T_r e_q^{-\beta(H-U_q)}. \tag{17}$$

It is then immediate to see that, for $q \to 1$, Eqs. (16) and (17) reproduce the conventional Boltzmann-Gibbs framework.

III. TWO-TIME GREEN's FUNCTIONS AND SPECTRAL DENSITY IN QUANTUM NONEXTENSIVE STATISTICAL MECHANICS

A. Definitions and spectral properties

In strict analogy with the extensive case [2,3,6], we define the two-time retarded ($\nu = r$) and advanced ($\nu = a$) q GFs in quantum nonextensive thermostatistics for two operators A and B as [44]

$$G_{qAB}^{(\nu)}(t,t') = -i\theta_\nu(t-t')\langle [A(t),B(t')]_\eta \rangle_q \equiv \langle\langle A(t);B(t')\rangle\rangle_q^{(\nu)} \quad (\nu = r,a). \tag{18}$$

Here, $\theta_r(t-t') = \theta(t-t')$, $\theta_a(t-t') = -\theta(t'-t)$, and $\theta(x)$ is the step function. In Eq. (18), $[\ldots,\ldots]_\eta$ denotes a commutator ($\eta = -1$) or an anticommutator ($\eta = +1$) and $X(t) = e^{iHt} X e^{-iHt}$ is the Heisenberg representation of the operator X, satisfying the Heisenberg equation of motion (EM) [45]

$$\frac{dX(t)}{dt} = i[H, X(t)]_-. \tag{19}$$

The definition (18) reproduces the conventional extensive formalism for $q = 1$ and allows us to develop the q GF framework equivalently with commutators or anticommutators. However, in practical calculations it will be convenient to use, in Eq. (18), $\eta = -1$ or $\eta = +1$ for bosonic or fermionic operators, respectively.

Physically relevant quantities, which enter the definition of q GFs, are the two time q correlation functions (CFs) $F_{qAB}(t,t') = \langle A(t)B(t')\rangle_q$ and $F_{qBA}(t',t) = \langle B(t')A(t)\rangle_q$ for the corresponding operators. Working within the equilibrium statistics, one can easily prove that the two-time q CFs and q GFs depend on times t and t' only through the difference $\tau = t - t'$. So one can write

$$G_{qAB}^{(\nu)}(t-t') = \langle\langle A(t-t');B\rangle\rangle_q^{(\nu)} = \langle\langle A;B(t'-t)\rangle\rangle_q^{(\nu)}. \tag{20}$$

This feature allows us to introduce the Fourier transforms

$$G_{qAB}^{(\nu)}(\tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G_{qAB}^{(\nu)}(\omega) e^{-i\omega\tau}, \tag{21}$$

$$F_{qXY}(\tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} F_{qXY}(\omega) e^{-i\omega\tau}, \tag{22}$$

where $G_{qAB}^{(\nu)}(\omega) = \langle\langle A(\tau);B\rangle\rangle_{q,\omega}^{(\nu)}$ will be named the ν-type q GF of A and B in the ω representation and $F_{qXY}(\omega) \equiv \langle X(\tau)Y\rangle_{q,\omega}$ will be called the q-spectral intensity of the time-dependent q CF $F_{qXY}(\tau)$, with $\mathfrak{I}(\omega) = \int_{-\infty}^{+\infty} d\tau \mathfrak{I}(\tau) e^{i\omega\tau}$ ($\mathfrak{I} = G_{qAB}^{(\nu)}, F_{qAB}, \ldots$).

We now define the time-dependent q SD for the operators A and B [2,3,7] as

$$\Lambda_{qAB}(\tau) = \langle [A(\tau), B]_\eta \rangle_q. \tag{23}$$

For its Fourier transform (the q SD in the ω representation)

$$\Lambda_{qAB}(\omega) = \langle [A(\tau), B]_\eta \rangle_{q,\omega}, \tag{24}$$

one immediately finds the exact result

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Lambda_{qAB}(\omega) = \langle [A,B]_\eta \rangle_q, \tag{25}$$

which constitutes an important "sum rule" for the q SD $\Lambda_{qAB}(\omega)$ to be used for physical consistency of practical calculations and approximations.

Then, using the integral representation for the ν-step function

$$\theta_\nu(\tau) = i \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \frac{e^{-ix\tau}}{x + (-1)^\nu i\varepsilon}, \quad \varepsilon \to 0^+, \tag{26}$$

where the symbol $(-1)^\nu$ stands for $+1$ if $\nu = r$ and -1 if $\nu = a$, we obtain the q-spectral representation

$$G^{(\nu)}_{qAB}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\Lambda_{qAB}(\omega')}{\omega - \omega' + (-1)^\nu i\varepsilon}, \quad \varepsilon \to 0^+, \tag{27}$$

for the Fourier transforms of the two-time q GFs (20).

Combining Eq. (27) and the sum rule (25), one can get another general result which, as in the extensive case, may play a relevant role in practical calculations involving the q GFs. Indeed, as $|\omega| \to \infty$ we have

$$\begin{aligned} G^{(\nu)}_{qAB}(\omega) &= \omega^{-1} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\Lambda_{qAB}(\omega')}{1 - \frac{\omega' - (-1)^\nu i\varepsilon}{\omega}} \\ &\approx \frac{\langle [A,B]_\eta \rangle_q}{\omega} + \frac{1}{\omega^2} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \Lambda_{qAB}(\omega')[\omega' - (-1)^\nu i\varepsilon] \\ &\quad + O\left(\frac{1}{\omega^3}\right) \end{aligned} \tag{28}$$

and hence

$$G^{(\nu)}_{qAB}(\omega) \sim \begin{cases} \omega^{-1} & \text{if } \langle [A,B]_\eta \rangle_q \neq 0, \\ \omega^{-\alpha} \quad (\alpha \geq 2) & \text{if } \langle [A,B]_\eta \rangle_q = 0, \end{cases} \tag{29}$$

which provides a relevant boundary condition for the q GFs.

As in the quantum extensive case [2,3] one can easily show that the q GFs $G^{(\nu)}_{qAB}(\omega)$, analytically continued in the complex ω plane, are analytical functions in the upper (for $\nu = r$) and lower (for $\nu = a$) half planes. Then, these functions can be combined to construct the q GF of complex ω:

$$G_{qAB}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\Lambda_{qAB}(\omega)}{\omega - \omega'} = \begin{cases} G^{(r)}_{qAB}(\omega), & \text{Im } \omega > 0, \\ G^{(a)}_{qAB}(\omega), & \text{Im } \omega < 0, \end{cases} \tag{30}$$

which is analytical in the whole complex ω plane with a cut along the real axis where singularities may occur.

It is worth noting that, in terms of the SD, no formal differences exist for the-spectral representations of the GFs in the extensive and nonextensive contexts. Hence, most of the developments in the extensive two-time GF framework remain formally valid in the nonextensive one. For instance, using the δ-function representation

$$\delta(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \left\{ \frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right\}, \tag{31}$$

we have the relation

$$\begin{aligned} \Lambda_{qAB}(\omega) &= i[G_{qAB}(\omega + i\varepsilon) - G_{qAB}(\omega - i\varepsilon)] \\ &= i[G^{(r)}_{qAB}(\omega) - G^{(a)}_{qAB}(\omega)], \end{aligned} \tag{32}$$

which expresses the q SD in terms of the related two-time q GFs in the ω representation. This result, which is expected to play an important role in the applications of the q GF method (as happens in the extensive case), suggests also that the cut for $G_{qAB}(\omega)$ along the real axis is determined by Eq. (32) and its singularities are given by real ω values satisfying the condition $\Lambda_{qAB}(\omega) \neq 0$. It is worth noting that, in general, $G_{qAB}(\omega)$ has to be regarded as a many-valued function of the complex variable ω. Hence, its singularities lie on the real axis on the first Riemann sheet. On the other sheets, the singularities may shift to the complex plane, leading to the appearance of complex poles.

Another important result can be easily obtained assuming $\Lambda_{qAB}(\omega)$ real and using the relation

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} d\omega' \frac{f(\omega')}{\omega' - \omega + (-1)^\nu i\varepsilon} \\ &\quad = \mathcal{P} \int_{-\infty}^{+\infty} d\omega' \frac{f(\omega')}{\omega' - \omega} - (-1)^\nu i\pi f(\omega), \end{aligned} \tag{33}$$

where $\mathcal{P}$ denotes the main part of the integral. From Eq. (27), we obtain indeed

$$G^{(\nu)}_{qAB}(\omega) = -\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{\Lambda_{qAB}(\omega')}{\omega' - \omega} - \frac{(-1)^\nu}{2} i\Lambda_{qAB}(\omega). \tag{34}$$

Hence we get

$$\mathrm{Re} G^{(\nu)}_{qAB}(\omega) = (-1)^\nu \mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi} \frac{\mathrm{Im} G^{(\nu)}_{qAB}(\omega')}{\omega' - \omega}, \tag{35}$$

with

$$\Lambda_{qAB}(\omega) = -2(-1)^\nu \mathrm{Im} G^{(\nu)}_{qAB}(\omega) \tag{36}$$

and, in particular,

$$\Lambda_{qAB}(\omega) = -2\mathrm{Im} G^{(r)}_{qAB}(\omega). \tag{37}$$

In analogy with the extensive counterparts [2,3], the relations (35) between the real and imaginary parts of $G^{(r)}_{qAB}(\omega)$ and $G^{(a)}_{qAB}(\omega)$ will be called q dispersion relations or nonextensive Kramer-Kronig relations.

In the next subsection we will derive spectral decompositions for $\Lambda_{qAB}(\omega)$, $G_{qAB}(\omega)$, $F_{qAB}(\tau)$, and $F_{qAB}(\omega)$ which allow us to obtain information about the nature of the GF singularities and hence about the excitations in nonextensive quantum many-body systems.

B. q-spectral decompositions

Let $\{|n\rangle\}$ and $\{E_n\}$ be the selected eigenvectors and eigenvalues of the Hamiltonian H of a many-body system and assume that $\{|n\rangle\}$ is a complete orthonormal set of states. In this representation, the q SD $\Lambda_{qAB}(\omega)$, as given by the Fourier transform (24), can be written as

$$\Lambda_{qAB}(\omega) = \frac{2\pi}{\tilde{Z}_q}\sum_{n,m}[1-(1-q)\beta(E_n-U_q)]^{q/1-q} \times\left\{1+\eta\left[\frac{1-(1-q)\beta(E_m-U_q)}{1-(1-q)\beta(E_n-U_q)}\right]^{q/1-q}\right\} \times A_{nm}B_{mn}\delta(\omega-\omega_{mn}), \tag{38}$$

where use is made of the OLM canonical framework for calculation of the q averages. In Eq. (38),

$$\tilde{Z}_q = T_r\rho^q = \sum_n [1-(1-q)\beta(E_n-U_q)]^{q/1-q}, \tag{39}$$

$X_{nm}=\langle n|X|m\rangle$, and $\omega_{mn}=E_m-E_n$. Besides, the spectral representation (30) for $G_{qAB}(\omega)$ yields

$$G_{qAB}(\omega) = \frac{2\pi}{\tilde{Z}_q}\sum_{n,m}[1-(1-q)\beta(E_n-U_q)]^{q/1-q} \times\left\{1+\eta\left[\frac{1-(1-q)\beta(E_m-U_q)}{1-(1-q)\beta(E_n-U_q)}\right]^{q/1-q}\right\}\frac{A_{nm}B_{nm}}{\omega-\omega_{mn}}. \tag{40}$$

Analogously, for the two-time q CF $F_{qAB}(\tau)$ and its Fourier transform $F_{qAB}(\omega)$, one easily finds

$$F_{qAB}(\tau) = \frac{2\pi}{\tilde{Z}_q}\sum_{n,m}[1-(1-q)\beta(E_n-U_q)]^{q/1-q}A_{nm}B_{mn}e^{-i\omega_{mn}\tau} \tag{41}$$

and

$$F_{qAB}(\omega) = \frac{2\pi}{\tilde{Z}_q}\sum_{n,m}[1-(1-q)\beta(E_n-U_q)]^{q/1-q}A_{nm}B_{mn} \times\delta(\omega-\omega_{mn}), \tag{42}$$

with similar expressions for $F_{qBA}(\tau)$ and $F_{qBA}(\omega)$.

The comparison of the previous relations with the corresponding extensive ones [2,3] indicates that the Tsallis statistics does not modify the meaning of the GF singularities, but changes substantially the structure of the spectral weights with the introduction of a mixing of the energy levels which is absent in the extensive framework. Equations (40) and (41) suggest indeed that the real poles of $G_{qAB}(\omega)$—i.e., the frequencies ω_{mn}—which are related to the eigenvalues of the Hamiltonian, represent the frequency (energy) spectrum of undamped excitations [oscillations in time of $F_{qAB}(\tau)$] in the system. It is worth noting that, for a macroscopic system, the δ poles in the spectral representation (40) are expected to be lying infinitesimally close, therewith defining a continuous function $\Lambda_{AB}(\omega)$ for real ω. Then one can speculate that the excitation concept may work only under the basic assumption that the q SD exhibits some pronounced peaks whose widths have to be considered as a direct measure of the damping or the lifetime of excitations (or q quasiparticles, elementary or collective depending on the physical nature of the operators A and B). As mentioned before, this picture should be associated with the appearance of further complicated singularities of the q GF $G_{qAB}(\omega)$ which may occur in the ω complex plane on the Riemann sheet below the real axis where $G^{(r)}_{qAB}(\omega)$ is not an analytical function. Hence, in practical calculations, one must search for the complex poles of $G^{(r)}_{qAB}(\omega)$ very close to, but below, the real ω axis. For each of them, the real part will determine the frequency of the excitations (the q excitation dispersion relation) and the imaginary part will represent their damping or lifetime. In this scenario, $\Lambda_{qAB}(\omega)$ will result a superposition of quasi-Lorentian peaks, at characteristic frequencies, whose widths will represent the damping of the related excitations. Of course, if these widths reduce to or are zero under appropriate physical conditions, the q SD will be given by a superposition of δ functions, signaling the occurrence of undamped excitations in the system.

C. Expressions of the two-time q correlation functions in terms of the q-spectral density

In the previous section compact relations between $G^{(\nu)}_{qAB}(\omega)$ and $\Lambda_{qAB}(\omega)$ have been obtained in strict analogy with the quantum extensive counterpart [2,3]. We will show below that the peculiar nature of the Tsallis probability distribution prevents us from expressing the two-time q CFs $\langle A(\tau)B\rangle_q$ and $\langle BA(\tau)\rangle_q$ directly in terms of the q SD $\Lambda_{qAB}(\omega)$ or the related q GFs $G^{(\nu)}_{qAB}(\omega)$ $(\nu=r,a)$.

First, it is worth recalling that in extensive quantum statistical mechanics it is a remarkable feature, the existence of direct relations, which allow us to express $\langle A(\tau)B\rangle = \langle A(\tau)B\rangle_{q=1}$ and $\langle BA(\tau)\rangle = \langle BA(\tau)\rangle_{q=1}$ in terms of $\Lambda_{AB}(\omega)$ or $G^{(\nu)}_{AB}(\omega)$ $(\nu=r,a)$ for two arbitrary operators A and B. For future utility we remember that these relations read [2,3]

$$\langle A(\tau)B\rangle = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{AB}(\omega)e^{-i\omega\tau}}{1+\eta e^{-\beta\omega}} \tag{43}$$

and

$$\langle BA(\tau)\rangle = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{AB}(\omega)e^{-i\omega\tau}}{e^{\beta\omega}+\eta}, \tag{44}$$

from which the corresponding static CFs can be immediately obtained setting $\tau=0$.

The situation becomes sensibly more complicated within the nonextensive context. To see this in a transparent way, it is convenient to start with the static q CFs for two arbitrary operators. Taking properly into account the presence of the δ functions in the spectral decomposition (38) for the q SD $\Lambda_{qAB}(\omega)$, we can also write

$$\frac{1}{2\pi}\frac{\Lambda_{qAB}(\omega)}{1+\eta\tilde{e}_q^{-\beta\omega}}=\tilde{Z}_q^{-1}\sum_{n,m}\tilde{e}_q^{-\beta(E_n-U_q)}\times\left\{\frac{1+\eta\tilde{e}_q^{-\beta(E_m-U_q)}/\tilde{e}_q^{-\beta(E_n-U_q)}}{1+\eta\tilde{e}_q^{-\beta(E_m-E_n)}}\right\}\times A_{nm}B_{mn}\delta(\omega-\omega_{mn}),\tag{45}$$

where we have conveniently introduced the modified q exponential function

$$\tilde{e}_q^x=[e_q^x]^q=[1+(1-q)x]^{q/1-q}.\tag{46}$$

Then, the integration over ω easily gives the exact relation

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)}{1+\eta\tilde{e}_q^{-\beta\omega}}=\tilde{Z}_q^{-1}\sum_{n,m}\tilde{e}_q^{-\beta(E_n-U_q)}\times\left\{\frac{1+\eta\tilde{e}_q^{-\beta(E_m-U_q)}/\tilde{e}_q^{-\beta(E_n-U_q)}}{1+\eta\tilde{e}_q^{-\beta(E_m-E_n)}}\right\}A_{nm}B_{mn},\tag{47}$$

with $\tilde{Z}_q=T_r\tilde{e}_q^{-\beta(H-U_q)}$.

By inspection of the right-hand side of this equation, one argues that it does not reduce to the correlation function $\langle AB\rangle_q$ as happens in the extensive case $q=1$. Nevertheless, if we introduce the "q operators" A_q and B_q (related to the original ones A and B) defined by the matrix elements

$$A_{qnm}=\langle n|A_q|m\rangle=C_q(n,m)A_{nm},\tag{48a}$$

$$B_{qnm}=\langle n|B_q|m\rangle=C_q(m,n)B_{nm},\tag{48b}$$

where

$$C_q(n,m)=\left\{\frac{1+\eta\tilde{e}_q^{-\beta(E_m-U_q)}/\tilde{e}_q^{-\beta(E_n-U_q)}}{1+\eta\tilde{e}_q^{-\beta(E_m-E_n)}}\right\}^{1/2},\tag{49}$$

with $C_{q=1}(n,m)=1$, Eq. (47) yields

$$\langle A_qB_q\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)}{1+\eta\tilde{e}_q^{-\beta\omega}}.\tag{50}$$

Similarly, with $\Lambda_{qBA}(\omega)=\eta\Lambda_{qAB}(-\omega)$, one finds

$$\langle B_qA_q\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)}{\tilde{e}_q^{\beta\omega}+\eta}.\tag{51}$$

The previous spectral relations, which express the q averages for products of the two q operators A_q and B_q in terms of the single q SD $\Lambda_{qAB}(\omega)$, are remarkably similar to the extensive static CFs for A and B and, as expected, they reduce consistently to them for $q\to 1$. Of course, it is $\langle X_qY_q\rangle_q\neq\langle XY\rangle_q$ $(X,Y=A,B)$ for $q\neq 1$.

Concerning the dynamical q CFs for q operators, using Eqs. (42), (45), and (49) it is now easy to show that the following relations are true:

$$\langle A_q(\tau)B_q\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)e^{-i\omega\tau}}{1+\eta\tilde{e}_q^{-\beta\omega}}\tag{52}$$

and

$$\langle B_qA_q(\tau)\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)e^{-i\omega\tau}}{\tilde{e}_q^{\beta\omega}+\eta},\tag{53}$$

which reduce to Eqs. (43) and (44) as $q\to 1$.

The substantial difference with respect to the case $q=1$ lies in the unfortunate feature that, for $q\neq 1$, the previous compact relations express the q averages for products of the complicated q operators A_q and B_q in terms of $\Lambda_{qAB}(\omega)$. This may constitute a serious difficulty in exploring physical cases involving directly the CFs of the operators A and B which enter the definitions of $G_{qAB}^{(\nu)}$ and $\Lambda_{qAB}(\omega)$. Nevertheless, it is possible to obtain explicit relations, although cumbersome and in general not very handy, which relate the CFs for the physical operators A and B to those for the corresponding q operators. Indeed, from Eqs. (47), (50), and (51) it is immediate to show that, for the static case,

$$\langle AB\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)}{1+\eta\tilde{e}_q^{-\beta\omega}}-\tilde{Z}_q^{-1}\sum_{n,m}\tilde{e}_q^{-\beta(E_n-U_q)}D_q(n,m)A_{nm}B_{mn}\tag{54}$$

and

$$\langle BA\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)}{\tilde{e}_q^{\beta\omega}+\eta}-\tilde{Z}_q^{-1}\sum_{n,m}\tilde{e}_q^{-\beta(E_n-U_q)}D_q(n,m)B_{nm}A_{mn},\tag{55}$$

with $D_q(n,m)=C_q^2(n,m)-1\to 0$ as $q\to 1$. Similar expressions are true for dynamical CFs $\langle A(\tau)B\rangle_q$ and $\langle BA(\tau)\rangle_q$, which will involve exponential oscillations in time.

A comparison with the corresponding extensive ones (43) and (44) allows us to understand, in a transparent way, the nature of the deviations from the extensive limit $q=1$. It is worth noting that, under the condition

$$|(1-q)\beta(H-U_q)|\ll 1,\tag{56}$$

the previous relations simplify to

$$\langle A(\tau)B\rangle_q\simeq\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)e^{-i\omega\tau}}{1+\eta\tilde{e}_q^{-\beta\omega}}\tag{57}$$

and

$$\langle BA(\tau)\rangle_q\simeq\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)e^{-i\omega\tau}}{\tilde{e}_q^{\beta\omega}+\eta},\tag{58}$$

which become exact for $q=1$. Of course, the simplified static q CFs for A and B can be obtained setting $\tau=0$ in Eqs. (57) and (58). These equations may be conveniently used in practical calculations. Of course, without the restrictive condition (56), one must use the cumbersome Eqs. (54) and (55) and resort to numerical calculations taking properly into account the cutoff condition (14a) and (14b). It is worth noting that, although the condition (56) is certainly verified in the limit $q\to 1$, it can be also realized for q far from unity with suitable choices of the parameters β and E_n-U_q. However, in view of a still reduced number of applications [27,46], in

order to gain experience in using the q many body formalism developed before for more complex situations, it may be in any case useful to consider weak nonextensivity conditions with q close to unity.

D. Parametric representation for the two-time q Green's functions

As a conclusion of this section, we will present an alternative way to introduce the two-time GFs, the related SD, and the two-time CFs by using a parametric representation, suggested in Refs. [25,26] in the context of the Kadanoff-Baym formalism [4]. This representation allows to express the relevant q quantities in terms of appropriate parametric integrals involving the corresponding $q=1$ ones.

The aim is to clarify the statement given in the Introduction about the effectiveness of the method in practical calculations and to stress again the potentiality of our framework for a direct calculation of the q quantities of interest. The basic idea is to take $b=1-(1-q)\beta(H-U_q)$ and alternatively $z=1+1/(1-q)$, $z=1/(1-q)$, and $z=q/(1-q)$ in the contour integral representation

$$\Gamma^{-1}(z) = ib^{1-z}\int_C \frac{du}{2\pi}\exp(-ub)(-u)^{-z}, \tag{59}$$

with $b>0$ and Re $z>0$. Here C denotes the contour in the z complex plane which starts from $+\infty$ on the real axis, encircles the origin once counterclockwise, and returns to $+\infty$. With these ingredients, one can easily obtain [25,26] the following representation for $Z_q(\beta)$, $G^{(\nu)}_{qAB}(\tau;\beta)$, and $\langle A_q(\tau)B_q\rangle_q$ (here we need to explicit the β dependence)

$$Z_q(\beta) = \int_C du\, K_q^{(1)}(u) Z_1[-\beta u(1-q)], \tag{60}$$

$$G^{(\nu)}_{qAB}(\tau;\beta) = \int_C du\, K_q^{(2)}(u) Z_1[-\beta u(1-q)] G^{(\nu)}_{1AB} \times(\tau;-\beta u(1-q)), \tag{61}$$

and

$$\langle A_q(\tau)B_q\rangle_q = \int_C du\, K_q^{(3)}(u) Z_1[-\beta u(1-q)]\langle A(\tau)B\rangle_{1,-\beta u(1-q)}, \tag{62}$$

where

$$K_q^{(1)}(u) = \frac{i}{2\pi}\Gamma\left(\frac{2-q}{1-q}\right)e^{-u[1+(1-q)\beta U_q]}(-u)^{-(2-q/1-q)}, \tag{63}$$

$$K_q^{(2)}(u) = -\frac{(1-q)}{(Z_q)^q}K_q^{(1)}(u), \tag{64}$$

$$K_q^{(3)}(u) = \frac{1-q}{q}K_q^{(2)}(u), \tag{65}$$

and Z_1, $G^{(\nu)}_{1AB}$, and $\langle A(\tau)B\rangle_{1,\beta}$ denote the corresponding extensive quantities.

An analogous integral representation for the q SD $\Lambda_{qAB}(\omega;\beta)$ can be simply obtained by replacing, in the Fourier transform of Eq. (61), the relation (27) which connects $G^{(\nu)}_{qAB}(\omega;\beta)$ to $\Lambda_{qAB}(\omega;\beta)$ and, for $q=1$, $G^{(\nu)}_{1AB}(\omega;\beta)$ to the extensive spectral density $\Lambda_{1AB}(\omega;\beta)=\langle[A(\tau),B]_\eta\rangle_{1,\beta;\omega}$. One finds

$$\Lambda_{qAB}(\omega;\beta) = \int_C du\, K_q^{(2)} Z_1[-\beta u(1-q)]\Lambda_{1AB} \times(\omega;-\beta u(1-q)). \tag{66}$$

In view of the previous cumbersome (although elegant) parametric representation, in our opinion the method to reduce q many-body problems to the corresponding extensive ones does not appear, in general, convenient in practical calculations. The two main reasons are that (a) it involves the *a priori* explicit calculation of $(q=1)$-quantities which, except for a limited number of simple cases, requires a first step of more or less reliable approximations; (b) after that, one must calculate nontrivial contour integrals of complicated functions. In general, this may be a formidable problem which could require additional approximations.

Thus, we consider it worthy of interest to develop many-body methods for a direct study of the q properties overcoming the *a priori* knowledge of the related extensive ones. We introduce below the appropriate extensions of two well-known and powerful quantum many-body methods (for the classical framework, see Appendix A). Next, we will support our statement by means of a direct detailed study of the q-induced nonextensivity effects on the low-temperature properties of a nontrivial many-boson model. A preliminary study, along this direction, of a d-dimensional Heisenberg spin model with long-range interactions was performed in Ref. [46]. These are only two nontrivial many-body problems used by us to test the effectiveness of our suggestion to perform direct nonextensivity calculations within a genuine q many-body theory.

IV. METHODS FOR DIRECT CALCULATION OF THE TWO-TIME q GREEN's FUNCTIONS AND THE q-SPECTRAL DENSITY

In this section we present an extension to the nonextensive quantum statistical mechanics of two intrinsically nonperturbative methods in strict analogy to the extensive counterpart. These methods will be called the q EMM, for a direct calculation of the two-time q GFs, and the q SDM, for a direct calculation of the q SD. In principle, in view of the exact relations established in the previous section, both the methods are completely equivalent in the sense that they should give exactly the q GFs and the related q SDs. Nevertheless, previous experiences in quantum [7] and classical [9] extensive statistical mechanics suggest that, in practical calculations, the q SDM may have several advantages for making more systematic and controllable approximations.

A. q equations-of-motion method

By differentiating Eq. (20) with respect $\tau=t-t'$ we obtain the q EM for the q GFs in the τ representation,

$$i\frac{d}{d\tau}\langle\langle A(\tau);B\rangle\rangle_q^{(\nu)} = \delta(\tau)\langle[A,B]_\eta\rangle_q + \langle\langle[A(\tau),H]_-;B\rangle\rangle_q^{(\nu)}, \tag{67}$$

or in the ω representation (more convenient in practical calculations),

$$\omega\langle\langle A(\tau);B\rangle\rangle_{q,\omega}^{(\nu)} = \langle[A,B]_\eta\rangle_q + \langle\langle[A(\tau),H]_-;B\rangle\rangle_{q,\omega}^{(\nu)}. \tag{68}$$

Equations (67) and (68) are not closed because higher-order q GFs occur in the problem. Therefore, one needs to consider an additional EM for these new functions which is again not closed. By iteration of this procedure, we obtain an infinite hierarchy of coupled EMs of increasing order which can be written in a compact form as

$$i\frac{d}{d\tau}\langle\langle L_H^m A(\tau);B\rangle\rangle_q^{(\nu)} = \delta(\tau)\langle[L_H^m A,B]_\eta\rangle_q + \langle\langle L_H^{m+1}A(\tau);B\rangle\rangle_q^{(\nu)}$$

$$(m=0,1,2,\ldots) \tag{69}$$

or

$$\omega\langle\langle L_H^m A(\tau);B\rangle\rangle_{q,\omega}^{(\nu)} = \langle[L_H^m A,B]_\eta\rangle_q + \langle\langle L_H^{m+1}A(\tau);B\rangle\rangle_{q,\omega}^{(\nu)}$$

$$(m=0,1,2,\ldots) \tag{70}$$

in time and Fourier representation, respectively. Here, we have used the EM (19) in the form

$$i\frac{d}{d\tau}A(\tau) = L_H^1 A(\tau) = [A(\tau),H]_-, \tag{71}$$

and the operator L_H^m means $L_H^0 A = A$, $L_H^1 A = [A,H]_-$, $L_H^2 A = [[A,H]_-,H]_-$, and so on. Note that the chain of q EMs in the representation (69) or (70) is formally the same for different types of q GFs and hence one can eliminate the index ν when the physical context does not offer ambiguity.

To solve the chain of EMs in the form (69) and (70), we must add appropriate boundary conditions which, in the ω representation, can be identified with the asymptotic behaviors (29). Of course, although Eqs. (69) and (70) are exact, it is impossible to find a complete solution for interacting systems. In practical calculations one is forced to use decoupling procedures, and hence approximate methods, to reduce the infinite chain of coupled equations to a finite closed one which may be solved. However, in general, systematic and controllable decouplings are not easy to find and one must check the reliability of a given approximation by comparing the results with experiments, simulations, or other types of approaches. The q SDM, which will be the subject of the next subsection, should be more flexible in such direction as it happens in the extensive case [7].

B. q-spectral density method

By successive derivatives of $\Lambda_{qAB}(\tau)$ [Eq. (23)] with respect to τ and using the EM (71), we have

$$\frac{d^m}{d\tau^m}\Lambda_{qAB}(\tau) = (-i)^m\langle[L_H^m A(\tau),B]_\eta\rangle_q \quad (m=0,1,2,\ldots). \tag{72}$$

Then, taking the Fourier transform of Eq. (72) and setting $\tau=0$, integration over ω yields finally the infinite set of exact equations for $\Lambda_{qAB}(\omega)$:

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\omega^m\Lambda_{qAB}(\omega) = \langle[L_H^m A,B]_\eta\rangle_q = \langle[A,\mathcal{L}_H^m B]_\eta\rangle_q$$

$$(m=0,1,2,\ldots), \tag{73}$$

where the operator $\mathcal{L}_H^m$ means $\mathcal{L}_H^0 B = B$, $\mathcal{L}_H^1 B = [H,B]_-$, $\mathcal{L}_H^2 B = [H,[H,B]_-]_-$, and so on. The quantity on the left-hand side of Eq. (73) will be called the m moment of $\Lambda_{qAB}(\omega)$, and the relations (73) constitute an infinite set of exact moment equations (MEs) or sum rules for the q SD.

The infinite set (73) can be seen in a different way. Since the η commutators and hence the q expectation values involved on the right-hand side can be calculated, at least in principle, it is quite remarkable that the m moments of the q SD can be explicitly obtained without *a priori* knowledge of the function $\Lambda_{qAB}(\omega)$. This important result implies that the sequence of Eq. (73) represents a typical moment problem. Its solution would yield the unknown q SD and hence all the related quantities (q GFs, q CFs, and other observables). Unfortunately, also this problem cannot be solved exactly and one must look for approximate solutions along the lines specified below which constitute the key idea of the original SDM [7].

1. Polar ansatz

As suggested by the exact spectral decomposition (38), one seeks for an approximation of $\Lambda_{qAB}(\omega)$ as a finite sum of properly weighted δ functions of the form (*polar ansatz*)

$$\Lambda_{qAB}(\omega) = 2\pi\sum_{k=1}^{n}\lambda_{qAB}^{(k)}\delta(\omega-\omega_{qAB}^{(k)}), \tag{74}$$

where n is an integer number. The unknown parameters $\lambda_{qAB}^{(k)}$ and $\omega_{qAB}^{(k)}$, depending on the physical nature of the operators A and B, have to be determined as a solution of the finite set of $2n$ equations obtained by inserting expression (74) into the first $2n$ MEs, Eq. (73). Physically, the parameters $\omega_{qAB}^{(k)}$ play the role of effective eigenvalues of the Hamiltonian and each of them represents a real pole of $G_{qAB}(\omega)$ corresponding to a mode of undamped oscillations for the q CF $\langle A(\tau)B\rangle_q$ [see Eq. (41)].

2. Modified Gaussian ansatz

As outlined at the end of Sec. III B, there are physical situations where the damping of oscillations in the system under study may be relevant and hence the polar approximation (74) is inadequate. In these cases, the basic idea of the SDM, related to the moment problem (73), remains still valid, but it is necessary to choose a more appropriate functional structure for the q SD which allows one to determine

the modes of excitations in the system and their damping or lifetime. In the extensive context, a generalization of the SDM in this sense was first proposed by Nolting and Oles [47] for Fermi systems and by Campana *et al.* [48] for Bose and classical systems whose SDs are not positive definite in the whole range of ω. Following their key idea—to assure the convergency of the q SD moments at any order and to preserve the intrinsic physical character of $\Lambda_{qAB}(\omega)$—one can assume for the q SD the *modified Gaussian ansatz* [48]

$$\Lambda_{qAB}(\omega) = 2\pi(\tilde{e}_q^{\beta\omega} + \eta)\sum_{k=1}^{n} \frac{\lambda_{qAB}^{(k)}}{\sqrt{\pi\Gamma_{qAB}^{(k)}}} e^{-(\omega-\omega_{qAB}^{(k)})^2/\Gamma_{qAB}^{(k)}}. \tag{75}$$

Clearly, with the functional representation (75) for $\Lambda_{qAB}(\omega)$, the width of the peak in $\omega=\omega_{qAB}^{(k)}$ is related to the parameter $\Gamma_{qAB}^{(k)}$ and the lifetime of the excitations with frequency $\omega_{qAB}^{(k)}$ has to be identified with $\tau_{qAB}^{(k)}=\sqrt{\Gamma_{qAB}^{(k)}}$ under the condition $\Gamma_{qAB}^{(k)}/[\omega_{qAB}^{(k)}]^2 \ll 1$. The choice (75) is only motivated by the fact that it makes direct contact with the notation used in the literature for extensive problems [47,48] and, in view of previous experiences, it is expected to simplify the algebra also in explicit calculations about q-induced nonextensivity effects.

As in the q EMM, also in the q SDM the problem remains to close the truncated finite set of q-MEs arising from the polar ansatz (74) or the modified Gaussian ansatz (75). In any case, evaluation of the right-hand side of Eq. (73) should generally involve higher-order q SDs. Hence, higher-order moment problems should be considered and the difficulty of calculations will increase considerably. So, in order to solve self-consistently the finite set of q MEs, which arises from Eq. (73) using the ansatz (74) or (75), it is usually necessary to use some decoupling procedures and thus to introduce, in a systematic way, additional consistent approximations in the SDM as in the extensive case [7,9,47,48].

V. NONEXTENSIVITY EFFECTS FOR A HIGH-DENSITY BOSE GAS WITH STRONG ATTRACTION BETWEEN PARTICLES

A. Model

For practical and explicit calculations we consider here a nontrivial Bose model introduced several years ago by Babichenko [31] with the aim to explore the properties of a Bose system with strong attraction between the particles. In contrast with real situations, where the interparticle interaction potential is characterized by a hard repulsive core at short distances and a strong attractive well at large distances, in Ref. [31] the repulsive core was assumed to be soft. This simplification allowed us to study exactly the properties of the model at $T=0$, making it possible to use diagrammatic techniques quite similar to the ones used for high-density Bose systems with a Coulomb pair interaction [49]. It is worth noting that, although the choice of a soft repulsive core does not correspond to the real situation, the possibility to obtain exact results for such a model is undoubtedly of interest since it leads to a strongly compressed ground state strictly related to the peculiar relation between the parameters of the attractive and repulsive parts of the pair interaction potential. In this sense, the model may be considered as a complement to the well-studied weakly nonideal Bose gas.

The same high-density Bose model was further studied in Ref. [32] to explore the finite-temperature effects in the context of the extensive many-body theory. Specifically, the usual Bogoliubov approximation was proved to be valid for this model and the exact results by Babichenko [31] were simply reproduced. Besides, the low-temperature properties of the model were again achieved without employing cumbersome diagrammatic techniques.

On this ground, also as a further contribution to the general Bose-Einstein condensation (BEC) scenario, we apply the q-many-body methods of Sec. VI to investigate the q-induced nonextensivity effects on the low-temperature behavior of the relevant thermodynamic quantities for the Babichenko model [31].

As a first step, we present below the definition of the model. Working in the grand-canonical ensemble and with periodic boundary conditions, the Bose model of interest is described by the second-quantized Hamiltonian

$$\hat{\mathcal{H}} = \hat{H} - \mu\hat{N} = \int d^3r\, \hat{\psi}^\dagger(\mathbf{r})\left\{-\frac{\hbar^2\nabla^2}{2m} - \mu\right\}\hat{\psi}(\mathbf{r}) + \frac{1}{2}\int d^3r \int d^3r'\, \varphi(|\mathbf{r}-\mathbf{r}'|)\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'), \tag{76}$$

where $\hat{N}$ is the total number operator of spinless bosons with mass m, μ is the chemical potential, and $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ are the usual Bose field operators. The pair interaction potential $\varphi(r)$ in (76) is assumed [31] as the superposition of a repulsive Yukawa-like potential of radius R_0 and of an attractive Gaussian well of radius $R>R_0$ and depth $U_0>0$, with the representation

$$\varphi(r) = \frac{\gamma}{r}\exp\left(-\frac{r}{R_0}\right) - U_0\exp\left(-\frac{r^2}{R^2}\right). \tag{77}$$

Here γ is a certain definite positive coupling parameter and the strong attraction or deep-well condition $U_0R^2 \gtrsim \hbar^2/m$ is assumed to be satisfied.

For our purposes it is convenient to choose the system of units with $\hbar=m=R_0=1$ and to work in the wave-vector $\{\mathbf{k}\}$ representation. So the grand-canonical Hamiltonian (76) assumes the form

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2V}\sum_{\{\mathbf{k}_\nu\}} \varphi(|\mathbf{k}_1-\mathbf{k}_3|)\,\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4} a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger a_{\mathbf{k}_3} a_{\mathbf{k}_4}, \tag{78}$$

where $\varepsilon_{\mathbf{k}}=k^2/2-\mu$, V is the volume of the system, and the Fourier transform $\varphi(k)$ of the pair interaction potential (77) is given by

$$\varphi(k) = \frac{\gamma}{1+k^2} - \pi^{3/2} U_0 R^3 \exp\left(-\frac{k^2 R^2}{4}\right), \tag{79}$$

with

$$\varphi(0) = \gamma - \pi^{3/2} U_0 R^3 = \pi^{3/2} \eta U_0 R^3, \quad 0 < \eta \ll 1, \tag{80}$$

assumed to be small.

With the previous definitions, the coupling parameters involved in the pair interaction potential are connected by the relation

$$\gamma = (1+\eta)\pi^{3/2} U_0 R^3, \tag{81}$$

and from the strong attraction condition $U_0 R^3 \gtrsim 1$, one easily finds that $\gamma \gtrsim 1$.

It is worth mentioning that, as shown in Ref. [31] and confirmed by the following calculations, the Gaussian form of the attractive part of $\varphi(r)$ is not essential. Indeed, the key condition to be used through the calculations for the second term in Eq. (77) is only a sufficiently smooth change with distance of $\varphi(r)$ such that its Fourier transform is localized in a small region of the wave-vector space ($k \lesssim 1/R$).

B. q equations-of-motion method within the Bogoliubov approximation

Adopting the conventional Bogoliubov approximation, usually restricted to weak interactions and low densities [50], but proved to be valid also for high-density charged Bose gas [51] and for the Babichenko high-density Bose model with strong attraction between the particles [32], the grand-canonical Hamiltonian (78) reduces to

$$\hat{\mathcal{H}} \simeq -\mu N_{q0} + \frac{1}{2}\frac{N_{q0}^2}{V}\varphi(0) + \sum_{\mathbf{k}\neq 0}\left[f_{q\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} h_{q\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}})\right], \tag{82}$$

where N_{q0} denotes an unknown q mean number of bosons in the condensate and

$$f_{q\mathbf{k}} = \varepsilon_{\mathbf{k}} + n_{q0}[\varphi(0) + \varphi(\mathbf{k})], \tag{83}$$

$$h_{q\mathbf{k}} = n_{q0}\varphi(\mathbf{k}), \quad n_{q0} = \frac{N_{q0}}{V}. \tag{84}$$

According to the standard Bogoliubov picture, one now should diagonalize the truncated Hamiltonian (82) by a linear canonical transformation and then proceed, at least in principle, to calculate the OLM low-temperature q-thermodynamic quantities with the prescription outlined in Sec. II (for the extensive case, see Ref. [32]). However, we find it convenient to follow a different approach which makes direct contact with the q many-body formalism developed in Secs. III–VI and allows us to explore, via direct explicit calculations, the q-induced nonextensivity in the low-temperature regime in a simpler and more transparent way.

Let us introduce the single-particle retarded and advanced two-time q GFs

$$G_{q\mathbf{k}}^{(\nu)}(\tau) = -i\theta_\nu(\tau)\langle[a_{\mathbf{k}}(\tau), a_{\mathbf{k}}^\dagger]_-\rangle_q, \tag{85}$$

$$\bar{G}_{q\mathbf{k}}^{(\nu)}(\tau) = -i\theta_\nu(\tau)\langle[a_{-\mathbf{k}}^\dagger(\tau), a_{\mathbf{k}}^\dagger]_-\rangle_q, \tag{86}$$

where the Heisenberg representation of the operators and the q averages have to be considered with respect to the truncated Hamiltonian (82).

Working in ω space, for the retarded or advanced q GFs $G_{q\mathbf{k}}(\omega) = \langle\langle a_{\mathbf{k}}(\tau); a_{\mathbf{k}}^\dagger\rangle\rangle_{q,\omega}$ and $\bar{G}_{q\mathbf{k}}(\omega) = \langle\langle a_{-\mathbf{k}}^\dagger(\tau); a_{\mathbf{k}}^\dagger\rangle\rangle_{q,\omega}$, one easily finds the two coupled EMs

$$\omega G_{q\mathbf{k}}(\omega) = 1 + f_{q\mathbf{k}} G_{q\mathbf{k}}(\omega) + h_{q\mathbf{k}} \bar{G}_{q\mathbf{k}}(\omega), \tag{87a}$$

$$\omega \bar{G}_{q\mathbf{k}}(\omega) = -h_{q\mathbf{k}} G_{q\mathbf{k}}(\omega) - f_{q\mathbf{k}} \bar{G}_{q\mathbf{k}}(\omega). \tag{87b}$$

This algebraic system can be simply solved to find

$$G_{q\mathbf{k}}(\omega) = \frac{\omega + f_{q\mathbf{k}}}{\omega^2 - \omega_{q\mathbf{k}}^2} = \frac{1}{2}\left[\left(1 + \frac{f_{q\mathbf{k}}}{\omega_{q\mathbf{k}}}\right)\frac{1}{\omega - \omega_{q\mathbf{k}}} + \left(1 - \frac{f_{q\mathbf{k}}}{\omega_{q\mathbf{k}}}\right)\frac{1}{\omega + \omega_{q\mathbf{k}}}\right], \tag{88}$$

$$\bar{G}_{q\mathbf{k}}(\omega) = \frac{-h_{q\mathbf{k}}}{\omega^2 - \omega_{q\mathbf{k}}^2} = \frac{-h_{q\mathbf{k}}}{2\omega_{q\mathbf{k}}}\left(\frac{1}{\omega - \omega_{q\mathbf{k}}} - \frac{1}{\omega + \omega_{q\mathbf{k}}}\right). \tag{89}$$

Here, the quantity

$$\omega_{q\mathbf{k}} = (f_{q\mathbf{k}}^2 - h_{q\mathbf{k}}^2)^{1/2} = \left\{\left[\frac{k^2}{2} - \mu + n_{q0}[\varphi(0) + \varphi(k)]\right]^2 - n_{q0}^2\varphi^2(k)\right\}^{1/2} \tag{90}$$

represents the energy (frequency) spectrum of the undamped elementary excitations in the system. To obtain the relevant q physical quantities it is now convenient to introduce the single-particle q SDs for the two q GFs $G_{q\mathbf{k}}$ and $\bar{G}_{q\mathbf{k}}$ defined as [see Eqs. (23) and (24)]

$$\Lambda_{q\mathbf{k}}(\omega) = \langle[a_{\mathbf{k}}(\tau), a_{\mathbf{k}}^\dagger]_-\rangle_{q,\omega} \tag{91}$$

and

$$\bar{\Lambda}_{q\mathbf{k}}(\omega) = \langle[a_{-\mathbf{k}}^\dagger(\tau), a_{\mathbf{k}}^\dagger]_-\rangle_{q,\omega}, \tag{92}$$

respectively.

Then, from Eqs. (32), (88), and (89), we immediately have the two δ-function representations for the spectral densities:

$$\Lambda_{q\mathbf{k}}(\omega) = \pi[(1+\gamma_{q\mathbf{k}})\delta(\omega - \omega_{q\mathbf{k}}) + (1-\gamma_{q\mathbf{k}})\delta(\omega + \omega_{q\mathbf{k}})] \tag{93}$$

and

$$\bar{\Lambda}_{q\mathbf{k}}(\omega) = \pi\lambda_{q\mathbf{k}}[\delta(\omega + \omega_{q\mathbf{k}}) - \delta(\omega - \omega_{q\mathbf{k}})], \tag{94}$$

where

$$\gamma_{q\mathbf{k}} = \frac{f_{q\mathbf{k}}}{\omega_{q\mathbf{k}}}, \quad \lambda_{q\mathbf{k}} = \frac{h_{q\mathbf{k}}}{\omega_{q\mathbf{k}}}. \tag{95}$$

We are now in the position to formally determine the relevant q-thermodynamic quantities simply taking into account the general relations stated in Sec. III.

First, we focus on the chemical potential that, within the Bogoliubov scenario, is defined by [52] $\mu = \langle \partial \hat{H} / \partial N_{q0} \rangle_q$. Since $\hat{H} = \hat{\mathcal{H}} + \mu \hat{N}$, with $\hat{N} \simeq N_{q0} + \Sigma_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$, Eq. (82) yields

$$\hat{H} \simeq \mu \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\frac{N_{q0}^2}{V}\varphi(0) + \sum_{\mathbf{k}\neq 0}\left[f_{q\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} h_{q\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}}) \right]. \tag{96}$$

Then, with simple algebra, we get

$$\mu = n_{q0}\varphi(0) + \frac{1}{n_{q0} V} \sum_{\mathbf{k}\neq 0} [(f_{q\mathbf{k}} - \varepsilon_{\mathbf{k}}) \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q + h_{q\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \rangle_q]. \tag{97}$$

Besides, for the q internal energy $U_q = \langle \hat{H} \rangle_q$, we find

$$U_q = \mu(N - N_{q0}) + \frac{1}{2}\frac{N_{q0}^2}{V}\varphi(0) + \sum_{\mathbf{k}\neq 0} [f_{q\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q + h_{q\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \rangle_q]. \tag{98}$$

In Eq. (98), $N = \langle \hat{N} \rangle_q$ and $N - N_{q0} = \Sigma_{\mathbf{k}\neq 0} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q$, where N is the total number of bosons in the system. In particular, the q depletion of condensate will be given by

$$n - n_{q0} = \frac{1}{V} \sum_{\mathbf{k}\neq 0} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q, \tag{99}$$

where $n = N/V$ is the total number density of particles. Notice that, using Eq. (99), expressions (97) and (98) for μ and $\langle \hat{H} \rangle_q$ can be also written in the most compact form

$$\mu = n\varphi(0) + \frac{1}{V} \sum_{\mathbf{k}\neq 0} \varphi(k) [\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q + \langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \rangle_q] \tag{100}$$

and

$$\langle \hat{N} \rangle_q = N_{q0}[\mu - n_{q0}\varphi(0)] + \frac{1}{2}\frac{N_{q0}^2 \varphi(0)}{V} + \frac{1}{2} \sum_{\mathbf{k}\neq 0} k^2 \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q. \tag{101}$$

Due to the complicated relations (54) and (55), it is not easy, in general, to calculate explicitly the q averages which enter Eqs. (97)–(99) in terms of the q SDs $\Lambda_{q\mathbf{k}}(\omega)$ and $\bar{\Lambda}_{q\mathbf{k}}(\omega)$. Intricate numerical calculations become necessary, or one must resort to reliable approximations to obtain analytical results. The problem becomes sensibly handier under the condition [see Eq. (56)]

$$|(1-q)\beta(\hat{\mathcal{H}} - \mathcal{U}_q)| \ll 1, \tag{102}$$

with $\mathcal{U}_q = U_q - \mu \langle \hat{N} \rangle_q$. Under this condition, from Eq. (57) it immediately follows that

$$\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{q\mathbf{k}}(\omega)}{\tilde{e}_q^{\beta\omega} - 1} \tag{103}$$

and

$$\langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \rangle_q = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\bar{\Lambda}_{q\mathbf{k}}(\omega)}{\tilde{e}_q^{\beta\omega} - 1}. \tag{104}$$

On the other hand, in the present Bogoliubov scenario, $\Lambda_{q\mathbf{k}}(\omega)$ and $\bar{\Lambda}_{q\mathbf{k}}(\omega)$ are exactly expressed by Eqs. (93) and (94). Then, from Eqs. (103) and (104) we easily find

$$\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle_q = \frac{1}{2}\left\{ \frac{1 + \gamma_{q\mathbf{k}}}{\tilde{e}_q^{\beta\omega_{q\mathbf{k}}} - 1} + \frac{1 - \gamma_{q\mathbf{k}}}{\tilde{e}_q^{-\beta\omega_{q\mathbf{k}}} - 1} \right\} \tag{105}$$

and

$$\langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \rangle_q = \frac{1}{2}\lambda_{q\mathbf{k}}\left\{ \frac{1}{\tilde{e}_q^{-\beta\omega_{q\mathbf{k}}} - 1} - \frac{1}{\tilde{e}_q^{\beta\omega_{q\mathbf{k}}} - 1} \right\}, \tag{106}$$

where $\gamma_{q\mathbf{k}}$ and $\lambda_{q\mathbf{k}}$ are given by Eqs. (95).

To calculate explicitly and consistently the q-thermodynamic quantities as functions of the temperature T and the total number density n, one must now express the chemical potential as a function of T and n, performing the sums over $\mathbf{k}$ in the previous basic expressions. This is not an easy task since, as q, μ enters the problem in a cumbersome way through the energy spectrum $\omega_{q\mathbf{k}}$. However, it will be shown below that, in the physical regimes of interest (high-density and low-temperature regimes), one has $|\mu - \mu_{q0}|/\mu_{q0} \ll 1$, with $\mu_{q0} = n_{q0}\varphi(0)$, and, consistently, $(n - n_{q0})/n = (N - N_{q0})/N \ll 1$, as expected in a Bogoliubov approximation scenario. This key feature allows us to simplify sensibly the problem setting, as a good (first) approximation, $\mu \simeq \mu_{q0} = n_{q0}\varphi(0) = (n_{q0}\gamma)\eta/(1+\eta)$ and also $n_{q0} \simeq n$ in the expressions of $\varepsilon_{\mathbf{k}}$, $\omega_{q\mathbf{k}}$, and $f_{q\mathbf{k}}$, which enter the previous general relations for the relevant q-thermodynamic quantities. So, in particular, with $\mu \simeq \mu_{q0} + \Delta\mu$, the correction $\Delta\mu$ to the chemical potential will be given by the last two terms of Eq. (97) with μ replaced by $n\varphi(0)$. In all cases, for explicit calculations, the expressions of $\varepsilon_{\mathbf{k}}$, $\omega_{q\mathbf{k}}$ and $f_{q\mathbf{k}}$ in the summands can be approximated, to leading order, as

$$\varepsilon_{\mathbf{k}} \approx \frac{k^2}{2} - n\varphi(0), \tag{107}$$

$$\omega_{q\mathbf{k}} \approx \left[\frac{k^4}{4} + nk^2\varphi(\mathbf{k}) \right]^{1/2} \equiv \omega_{\mathbf{k}}, \tag{108}$$

$$f_{q\mathbf{k}} \approx \frac{k^2}{2} + n\varphi(\mathbf{k}) \equiv f_{\mathbf{k}}. \tag{109}$$

Notice that $\omega_{\mathbf{k}}$, on the right-hand side of expression (108), is just the Bogoliubov energy spectrum of elementary excitations.

After that, with $V^{-1}\Sigma_{\mathbf{k}\neq 0}(\cdots) \overset{V\to\infty}{\to} (2\pi^2)^{-1}\int_0^\infty dk\, k^2(\cdots)$, very complicated integrals remain again to be performed due to the presence of the power-law function $\tilde{e}_q^x$ involved in the integrands. Since resorting to numerical calculations by variation of β and q is beyond the purposes of the present work, to obtain demonstrative explicit results one is then compelled to make further suitable approximations as the known factorization procedure [53] or expansions close to the extensive value $q=1$ of the Tsallis parameter. We follow here the second direction because it is simple and instructive to estimate, to the leading order in $q-1$, the q nonextensive effects on the already known exact extensive results [32] in the high-density and low-temperature regimes where the integrals can be explicitly calculated.

To the first order in $q-1$ one obtains the expansion

$$\frac{1}{\tilde{e}_q^{\pm\beta\omega_\mathbf{k}}-1} \simeq \frac{1}{e^{\pm\beta\omega_\mathbf{k}}-1} + \frac{1}{2}(1-q) \times \frac{e^{\pm\beta\omega_\mathbf{k}}}{(e^{\pm\beta\omega_\mathbf{k}}-1)^2}[\beta^2\omega_\mathbf{k}^2 \pm 2\beta\omega_\mathbf{k}]. \quad (110)$$

So, for the basic q-thermodynamic quantities, we can formally write (with a cumbersome but simple algebra)

$$n-n_{q0} \simeq \frac{1}{2V}\sum_{\mathbf{k}\neq 0}\frac{f_\mathbf{k}-\omega_\mathbf{k}}{\omega_\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{f_\mathbf{k}}{\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1} + \frac{1}{2}(1-q) \times\left\{\frac{\beta^2}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}^2 e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2} + 2\frac{\beta}{V}\sum_{\mathbf{k}\neq 0}\frac{f_\mathbf{k} e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2}\right\}, \quad (111)$$

$$\mu \simeq n\varphi(0) + \frac{1}{n}\left[\frac{-1}{2V}\sum_{\mathbf{k}\neq 0}(f_\mathbf{k}-\omega_\mathbf{k}) + \frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}}{e^{\beta\omega_\mathbf{k}}-1}\right] - \frac{1}{2n}\left[\frac{1}{2V}\sum_{\mathbf{k}\neq 0}k^2\frac{f_\mathbf{k}-\omega_\mathbf{k}}{\omega_\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{k^2 f_\mathbf{k}}{\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1}\right] + \frac{1}{2}(1-q)\left\{\frac{\beta^2}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}^2\varphi(k)e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2} + 2\frac{\beta}{V}\sum_{\mathbf{k}\neq 0}\frac{k^2\varphi(k)e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2}\right\}, \quad (112)$$

$$\langle\hat{H}\rangle_q \simeq \frac{1}{2}\frac{N^2}{V}\varphi(0) - \frac{1}{2}\sum_{\mathbf{k}\neq 0}(f_\mathbf{k}-\omega_\mathbf{k}) + \sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}}{e^{\beta\omega_\mathbf{k}}-1} + \frac{1}{2}(1-q) \times\left\{\beta^2\sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}^2 f_\mathbf{k} e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2} + 2\beta\sum_{\mathbf{k}\neq 0}\frac{\omega_\mathbf{k}^2 e^{\beta\omega_\mathbf{k}}}{(e^{\beta\omega_\mathbf{k}}-1)^2}\right\}. \quad (113)$$

Now all the integrals in Eqs. (111)–(113) can be consistently calculated under the conditions $(n\gamma)^{1/2}\gg 1$ (high-density regime) and $\beta\gg 1$ (low-temperature regime) where it is reasonable to speculate that the Bogoliubov approximation preserves its validity [32]. Indeed, making the transformation $x=k(n\gamma)^{1/4}$, one sees that the principal contribution to the integrals is made by $x\sim 1$ so that neglecting $O((n\gamma)^{-1/2})$ or exponentially small terms is quite legitimate [31]. A similar situation arises also in a high-density Bose gas with Coulomb interaction between bosons [49,51]. Bearing this in mind, in the high-density and low-temperature limits, for the depletion of condensate, the chemical potential, and the q internal energy density $u_q=U_q/V=\langle\hat{H}\rangle_q/V$ as functions of T and n, one finds (see Appendix B for calculations of the integrals)

$$n-n_{q0} \simeq a(n\gamma)^{3/4} + \frac{1}{12}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^2 + (1-q)\frac{1}{6}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^2 \times\left\{1+\frac{2\pi^2}{5}\left(\frac{\eta+1}{\eta}\right)(n\gamma)^{-1}T\right\}, \quad (114)$$

$$\mu \simeq \frac{\eta+1}{\eta}(n\gamma) - b\gamma(n\gamma)^{1/4} + \frac{\pi^2}{60}\left(\frac{\eta+1}{\eta}\right)^{3/2}\gamma(n\gamma)^{-5/2}T^4 + (1-q)\frac{\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{1/2}\gamma(n\gamma)^{-3/2}T^3 \times\left\{1+2\left(\frac{\eta+1}{\eta}\right)(n\gamma)^{-1}T\right\}, \quad (115)$$

$$u_q \simeq \frac{1}{2}\frac{\eta+1}{\eta}\frac{1}{\gamma}(n\gamma)^2 - \frac{4}{5}b(n\gamma)^{5/4} + \frac{\pi^2}{30}\left(\frac{\eta+1}{\eta}\right)^{3/2}(n\gamma)^{-3/2}T^4 + (1-q)\frac{\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^3 \times\left\{1+2\left(\frac{\eta+1}{\eta}\right)(n\gamma)^{-1}T\right\}, \quad (116)$$

where $a=\sqrt{2}\Gamma^2(1/4)/(48\pi^{5/2})$ and $b=\sqrt{2}\Gamma^2(3/4)/(4\pi^{5/2})$.

Notice that, from Eqs. (114) and (116), for $(n_{q0}\gamma)^{1/2} \simeq (n\gamma)^{1/2}\gg 1$ and $T\ll 1$ one has $(n-n_{q0})/n\ll 1$ and $|\mu-\mu_{q0}|/\mu_{q0}\ll 1$, as expected, signaling the internal consistency of calculations. Of course, for $q=1$ the extensive results are exactly reproduced [31,32]. From the previous basic relations one can now calculate all the q-thermodynamic quantities of the system using the OLM formalism [29,30]. For instance, the q specific heat at constant volume is given by

$$C_{qV}=\left(\frac{\partial u_q}{\partial T}\right)_V \simeq \frac{2\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{3/2}(n\gamma)^{-3/2}T^3 + (1-q) \times\frac{\pi^2}{5}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^2\left\{1+\frac{8}{3}\left(\frac{\eta+1}{\eta}\right)(n\gamma)^{-1}T\right\}. \quad (117)$$

It is worth noting that, as suggested by previous results, the q-induced thermal effects may compete sensibly with the extensive contributions.

C. q-spectral density method at work: The two-pole approximation

In this section we apply the q SDM directly to the grand-canonical Hamiltonian (78) by avoiding the Bogoliubov approximation and hence the known troubles related to a trun-

cated Hamiltonian which no longer conserves the particle number and to the unwanted dependence of N_{q0}.

Taking advantage from the previous results, we will work rather via a two-δ-function ansatz for the appropriate q SD as suggested by Eq. (93). Then, we will establish the conditions for which the results of Sec. V B can be reproduced in the high-density and low-temperature regimes. As we shall see, the procedure will also open a window towards a possible systematic study of the q-thermodynamics of other more realistic second-quantized many-boson models.

For the Hamiltonian (78) in the wave-vector representation, we introduce the single-particle q SD

$$\Lambda_{q\mathbf{k}}(\omega) = \langle [a_{\mathbf{k}}(\tau), a_{\mathbf{k}}^{\dagger}]_{-} \rangle_{q,\omega}. \tag{118}$$

Then, the general q MEs, Eq. (73), for the present problem can be written as

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega^m \Lambda_{q\mathbf{k}}(\omega) = \langle [L_{\hat{\mathcal{H}}}^m a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}]_{-} \rangle_q = \langle [a_{\mathbf{k}}, \mathcal{L}_{\hat{\mathcal{H}}}^m a_{\mathbf{k}}^{\dagger}]_{-} \rangle_q$$

$$(m = 0, 1, 2, \ldots). \tag{119}$$

Using the usual bosonic canonical commutation relations, tedious but simple algebra yields for $m=0,1,2$ (the case of interest for us)

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Lambda_{q\mathbf{k}}(\omega) = 1, \tag{120}$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega \Lambda_{q\mathbf{k}}(\omega) = \varepsilon_{\mathbf{k}} + \varphi(0) n + \frac{1}{V} \sum_{\mathbf{k}'} \varphi(|\mathbf{k} - \mathbf{k}'|) N_{q\mathbf{k}'}, \tag{121}$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega^2 \Lambda_{q\mathbf{k}}(\omega) = -\varepsilon_{\mathbf{k}}^2 + 2\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega \Lambda_{q\mathbf{k}}(\omega) + L_q + \frac{2}{V} \sum_{\mathbf{k}'} \varphi(|\mathbf{k} - \mathbf{k}'|) \lambda_{q\mathbf{k}'}, \tag{122}$$

where

$$n = \frac{1}{V} \sum_{\mathbf{k}} \langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \rangle_q, \tag{123}$$

$$N_{q\mathbf{k}} = \langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \rangle_q, \tag{124}$$

$$L_q = \frac{1}{V^2} \sum_{\{\mathbf{k}_\nu\}} \delta_{\mathbf{k}_1+\mathbf{k}_2;\mathbf{k}_3+\mathbf{k}_4} \varphi^2(|\mathbf{k}_1 - \mathbf{k}_3|) \langle a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_3} a_{\mathbf{k}_2}^{\dagger} a_{\mathbf{k}_4} \rangle_q, \tag{125}$$

and

$$\lambda_{q\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \varphi(|\mathbf{k} - \mathbf{k}'|) N_{q\mathbf{k}'} + \frac{1}{V} \sum_{\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4\}} \varphi(|\mathbf{k}_1 - \mathbf{k}|) \times \delta_{\mathbf{k}_1+\mathbf{k}_2;\mathbf{k}+\mathbf{k}_4} \langle a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2}^{\dagger} a_{\mathbf{k}} a_{\mathbf{k}_4} \rangle_q. \tag{126}$$

As we see, L_q and $\lambda_{q\mathbf{k}}$ introduce in the problem two-particle q CFs which should be expressed in terms of $\Lambda_{q\mathbf{k}}(\omega)$ to close the truncated system of MEs (120)–(122). Unfortunately, this constitutes a serious difficulty since, as shown in Sec. III, there is no simple relation between a CF of the type $\langle BA \rangle_q$ and the corresponding q SD $\Lambda_{qAB}(\omega)$. However, under the condition $|(1-q)\beta(\hat{\mathcal{H}} - \mathcal{U}_q)| \ll 1$, the calculations simplify sensibly. Indeed, in this case, one can assume $\langle BA \rangle_q \simeq (2\pi)^{-1} \int_{-\infty}^{+\infty} d\omega \, \Lambda_{qAB}(\omega) / (\tilde{e}_q^{\beta\omega} - 1)$ and in Eqs. (121)–(124) and (126) one has

$$N_{q\mathbf{k}} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{q\mathbf{k}}(\omega)}{\tilde{e}_q^{\beta\omega} - 1}. \tag{127}$$

Besides, with standard straightforward algebra [7,32], we have also

$$\frac{1}{V} \sum_{\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4\}} \varphi(|\mathbf{k}_1 - \mathbf{k}|) \delta_{\mathbf{k}_1+\mathbf{k}_2;\mathbf{k}+\mathbf{k}_4} \langle a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2}^{\dagger} a_{\mathbf{k}} a_{\mathbf{k}_4} \rangle_q = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\omega - \varepsilon_{\mathbf{k}}}{\tilde{e}_q^{\beta\omega} - 1} \Lambda_{q\mathbf{k}}(\omega). \tag{128}$$

This allows us to express $\lambda_{q\mathbf{k}}$ in Eq. (126) in terms of $\Lambda_{q\mathbf{k}}(\omega)$ as

$$\lambda_{q\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \varphi(|\mathbf{k} - \mathbf{k}'|) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{q\mathbf{k}'}(\omega)}{\tilde{e}_q^{\beta\omega} - 1} + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\omega - \varepsilon_{\mathbf{k}}}{\tilde{e}_q^{\beta\omega} - 1} \Lambda_{q\mathbf{k}}(\omega) \tag{129}$$

and to obtain for the q internal energy the following expression in terms of the single-particle q SD:

$$U_q = \langle \hat{H} \rangle_q = \mu N + \frac{1}{2} \sum_{\mathbf{k}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\omega + \varepsilon_{\mathbf{k}}}{\tilde{e}_q^{\beta\omega} - 1} \Lambda_{q\mathbf{k}}(\omega), \tag{130}$$

where $N = \Sigma_{\mathbf{k}} N_{q\mathbf{k}}$.

Nevertheless, the quantity L_q cannot be expressed exactly in terms of $\Lambda_{q\mathbf{k}}(\omega)$, so that the truncated system (120)–(122) is not yet closed and one must resort to additional decoupling procedures [7] which allow one to express also L_q in terms of $\Lambda_{q\mathbf{k}}(\omega)$. We shall see below that this problem can be simply solved on physical grounds in the regime of interest.

Working within the spirit of the SDM [7,32] (see Sec. III), we assume for $\Lambda_{q\mathbf{k}}(\omega)$ the two-δ-function ansatz [also suggested by Eq. (93)]

$$\Lambda_{q\mathbf{k}}(\omega) = \pi [(1 + \gamma_{q\mathbf{k}}) \delta(\omega - \omega_{q\mathbf{k}}) + (1 - \gamma_{q\mathbf{k}}) \delta(\omega + \omega_{q\mathbf{k}})]. \tag{131}$$

Then, with some algebra, Eqs. (120)–(122) yield, for the unknown functional parameters $\gamma_{q\mathbf{k}}$ and $\omega_{q\mathbf{k}}$, the self-consistent equations

$$\gamma_{q\mathbf{k}}\omega_{q\mathbf{k}} = \varepsilon_{\mathbf{k}} + \varphi(0)n + \frac{1}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)N_{q\mathbf{k}'}, \quad (132)$$

$$\omega_{q\mathbf{k}}^2 = L_q + \frac{2}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)\lambda_{q\mathbf{k}'} + \varepsilon_{\mathbf{k}}(\varepsilon_{\mathbf{k}} + 2n\varphi(0)) + \frac{2}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)N_{q\mathbf{k}'}. \quad (133)$$

In these equations $N_{q\mathbf{k}}$ is formally given by Eq. (105) and

$$\lambda_{q\mathbf{k}} = \frac{1}{4V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)\left\{\frac{1+\gamma_{q\mathbf{k}'}}{\tilde{e}_q^{\beta\omega_{q\mathbf{k}'}}-1} + \frac{1-\gamma_{q\mathbf{k}'}}{\tilde{e}_q^{-\beta\omega_{q\mathbf{k}'}}-1}\right\} + \frac{1}{2}\left\{\frac{(\omega_{q\mathbf{k}}-\varepsilon_{\mathbf{k}})(1+\gamma_{q\mathbf{k}})}{\tilde{e}_q^{\beta\omega_{q\mathbf{k}}}-1} - \frac{(\omega_{q\mathbf{k}}+\varepsilon_{\mathbf{k}})(1-\gamma_{q\mathbf{k}})}{\tilde{e}_q^{-\beta\omega_{q\mathbf{k}}}-1}\right\}. \quad (134)$$

Of course, when $\gamma_{q\mathbf{k}}$ and $\omega_{q\mathbf{k}}$ are known, for the q GF related to $\Lambda_{q\mathbf{k}}(\omega)$,

$$G_{q\mathbf{k}}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\frac{\Lambda_{q\mathbf{k}}(\omega')}{\omega-\omega'}, \quad (135)$$

we will have

$$G_{q\mathbf{k}}(\omega) = \frac{1}{2}\left\{\frac{1+\gamma_{q\mathbf{k}}}{\omega-\omega_{q\mathbf{k}}} + \frac{1-\gamma_{q\mathbf{k}}}{\omega+\omega_{q\mathbf{k}}}\right\} = \frac{\omega+\varepsilon_{\mathbf{k}}+n\varphi(0)+\frac{1}{V}\sum_{\mathbf{k}'}\varphi(|\mathbf{k}-\mathbf{k}'|)N_{q\mathbf{k}'}}{\omega^2-\omega_{q\mathbf{k}}^2}. \quad (136)$$

As expected, Eq. (136) implies two poles $\omega = \pm\omega_{q\mathbf{k}}$ (with $\omega_{q\mathbf{k}} \geq 0$) on the ω real axis and hence the parameter $\omega_{q\mathbf{k}}$ determines the q energy spectrum of the undamped elementary excitations in the system, formally given by

$$\omega_{q\mathbf{k}}^2 = \left\{L_q + \frac{2}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)\lambda_{q\mathbf{k}'} + \varepsilon_{\mathbf{k}}\left(\varepsilon_{\mathbf{k}} + 2n\varphi(0) + \frac{2}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)N_{q\mathbf{k}'}\right)\right\}^{1/2}. \quad (137)$$

In the polar ansatz (131), to determine $\Lambda_{q\mathbf{k}}(\omega)$ and hence all the relevant q-thermodynamic quantities, the problem remains to express the unknown higher-order quantity L_q in terms of the parameters $\gamma_{q\mathbf{k}}$ and $\omega_{q\mathbf{k}}$.

This difficulty can be easily overcome if we limit ourselves to explore the condensate phase (which is of main interest for our Bose model in the high-density and low-T limits) when a macroscopic population of the zero-moment single-particle state takes place. Indeed, in this situation one can obtain a formally exact expression for L_q setting $\omega_{q\mathbf{k}=0} = 0$ below a certain critical temperature [1,54], to find, from Eq. (137),

$$L_q = -\mu^2 + \mu\left(2\varphi(0)n + \frac{2}{V}\sum_{\mathbf{k}'} \varphi(k')N_{q\mathbf{k}'}\right) - \frac{2}{V}\sum_{\mathbf{k}'} \varphi(k')\lambda_{q\mathbf{k}'}, \quad (138)$$

where $N_{q\mathbf{k}}$ and $\lambda_{q\mathbf{k}}$ are given by Eqs. (127) and (129).

Then, in the condensate state the q energy spectrum of the elementary excitations can be written in the form

$$\omega_{q\mathbf{k}} = \left\{\frac{k^4}{4} + k^2\left[\frac{1}{V}\sum_{\mathbf{k}'} \varphi(|\mathbf{k}-\mathbf{k}'|)N_{q\mathbf{k}'} - [\mu - n\varphi(0)]\right] + \Omega_{q\mathbf{k}}\right\}^{1/2}, \quad (139)$$

where

$$\Omega_{q\mathbf{k}} = \frac{2}{V}\sum_{\mathbf{k}'} [\varphi(|\mathbf{k}-\mathbf{k}'|) - \varphi(|\mathbf{k}'|)](\lambda_{q\mathbf{k}'} - \mu N_{q\mathbf{k}'}). \quad (140)$$

Of course, Eq. (139) does not provide the explicit q dispersion relation as a function of T and n since $N_{q\mathbf{k}}$ and $\lambda_{q\mathbf{k}}$ contain $\omega_{q\mathbf{k}}$ itself and $\gamma_{q\mathbf{k}}$ which have to be still determined as solutions of the closed system of equations (132) and (139).

In general, this problem is difficult to solve and, as for the extensive counterpart [7,55], one is forced to consider asymptotic regimes for obtaining explicit results or to use numerical calculations. Obviously, solving numerically the previous closed self-consistent set of coupled q MEs, by variation of T and q, constitutes a formidable *tour de force* (beyond the purposes of the present paper) which requires a separate study. Here, to avoid obscuring complicated calculations, we follow the procedure already used for the extensive case [55] to obtain, in a natural way, predictions for the condensate state assuming

$$\frac{N-N_{q0}}{N} = \frac{n-n_{q0}}{n} \ll 1 \quad (141)$$

and

$$\frac{|\Delta\mu|}{n\varphi(0)} \ll 1, \quad (142)$$

where $\Delta\mu = \mu - n\varphi(0)$.

These inequalities, already used in the previous subsection just as the basic conditions for the validity of the Bogoliubov approximation, must be consistently satisfied at the end of calculations.

It is easy to check that under conditions (141) and (142), with $\mu \simeq n\varphi(0)$ and $n_{q0} \simeq n$ to leading order, the q MEs simplify to [56]

$$\gamma_{q\mathbf{k}}\omega_{q\mathbf{k}} \approx k^2/2 + n\varphi(k), \quad \omega_{q\mathbf{k}}^2 \approx [\omega_{\mathbf{k}}^{(B)}]^2 + \Omega_{q\mathbf{k}}^{(0)}. \quad (143)$$

Here

$$\omega_{\mathbf{k}}^{(B)} = \left\{ \frac{k^4}{4} + nk^2\varphi(\mathbf{k}) \right\}^{1/2} \tag{144}$$

is the Bogoliubov spectrum and

$$\begin{aligned}\Omega_{q\mathbf{k}}^{(0)} &= \Omega_{q\mathbf{k}}\big|_{\mu\simeq n\varphi(0)} \\ &\simeq \frac{n}{V}\sum_{\mathbf{k}'} \varphi(k')[\varphi(|\mathbf{k}-\mathbf{k}'|) - \varphi(k')] \\ &\times\left\{ 1 + \left[\frac{1}{\tilde{e}_q^{\beta\omega_{\mathbf{k}'}^{(B)}} - 1} + \frac{1}{\tilde{e}_q^{-\beta\omega_{\mathbf{k}'}^{(B)}} - 1} \right] \right\} \\ &+ \frac{1}{V}\sum_{\mathbf{k}'} [\varphi(|\mathbf{k}-\mathbf{k}'|) - \varphi(k')] \left(\omega_{\mathbf{k}'}^{(B)} - \gamma_{\mathbf{k}'}\frac{k'}{2} \right) \\ &\times\left[\frac{1}{\tilde{e}_q^{\beta\omega_{\mathbf{k}'}^{(B)}} - 1} - \frac{1}{\tilde{e}_q^{-\beta\omega_{\mathbf{k}'}^{(B)}} - 1} \right]\end{aligned} \tag{145}$$

provides an estimate of the correction to the Bogoliubov spectrum where the inequality $|\Omega_{q\mathbf{k}}^{(0)}| \ll [\omega_{\mathbf{k}}^{(B)}]^2$ has to be checked at the end of calculations. Of course, from Eq. (143), we have for the parameter $\gamma_{q\mathbf{k}}$

$$\gamma_{q\mathbf{k}} \simeq \frac{k^2/2 + n\varphi(k)}{\omega_{\mathbf{k}}^{(B)}} \left\{ 1 - \frac{\Omega_{q\mathbf{k}}^{(0)}}{2[\omega_{\mathbf{k}}^{(B)}]^2} \right\}. \tag{146}$$

Thus, the original q-moment problem is solved and all the q-thermodynamic properties for the condensate region under the conditions (141) and (142), and in particular the q depletion of the condensate $n - n_{q0} = \Sigma_{\mathbf{k}\neq 0} N_{q\mathbf{k}}$, follow immediately from the spectral relations (127), (130), and (135), with $\Lambda_{q\mathbf{k}}(\omega)$ given by the two-δ-function representation (131). All the relevant expressions in terms of $\omega_{q\mathbf{k}}$ and $\gamma_{q\mathbf{k}}$ at the Bogoliubov approximation level can be easily derived or also obtained from Ref. [55] with e^x replaced by $\tilde{e}_q^x$. These are quite cumbersome and not particularly instructive, and hence they will not be reported here for brevity reasons.

It is remarkable that, by a straightforward extension to the case $q \neq 1$ of the procedure used in Ref. [55] for the extensive case, one can systematically estimate, at least in principle, the corrections to the predictions of the Bogoliubov approximation for any plausible potential $\varphi(k)$.

For the high-density Bose model under study in the regime of interest, the parameters $\omega_{q\mathbf{k}}$ and $\gamma_{q\mathbf{k}}$ in all the summands can be replaced by their leading expressions $\omega_{\mathbf{k}}^{(B)}$ and $\gamma_{\mathbf{k}}^{(B)} = [k^2/2 + n\varphi(k)]/\omega_{\mathbf{k}}^{(B)}$ and the basic equations become essentially identical to that obtained in Sec. V B. In particular, with $(n\gamma)^{1/2} \gg 1$ and $\beta \gg 1$, all the results (114)–(117) near to the extensive regime are easily reproduced and the conditions (141) and (142) are found to be consistently satisfied.

As a conclusion, we make some comments which may be of practical interest for future calculations. For the high-density Bose model defined in SecV A, and possibly for other models, the q SDM offers some advantages with respect to the method used in Sec. V B based on the Bogoliubov scenario. Adopting the q SDM one avoids the *a priori* replacement $a_0^\dagger, a_0 \to N_{q0}^{1/2}$ in the original Hamiltonian and hence some consequent conceptual difficulties [57] in both the extensive and nonextensive cases. Besides, it allows one to obtain, in a systematic way, the corrections to the Bogoliubov predictions. Finally, the procedure may be adapted to other more realistic situations provided that the conditions (141) and (142) are consistently satisfied at the end of calculations.

VI. CONCLUDING REMARKS

In this paper we have extended the general formalism of two-time GFs [2,3,7] to nonextensive quantum statistical mechanics in the OLM representation [29]. Particular attention has been devoted to the spectral properties and to the concept of SDs, which is expected to play an important role in explicit calculations also in nonextensive quantum many-body problems. Besides, we have presented the q EMM and the nonextensive version of the less known SDM [7] for a direct calculation of the q GFs and q SDs, respectively. A remarkable feature is that these methods should allow one, at least in principle, to explore on the same footing the nonextensivity effects for a wide variety of quantum many-body systems overcoming the *a priori* knowledge of the q partition function and hence of the q free energy. Unfortunately, in contrast to the extensive case [2,3,7], the q CFs of two generic operators A and B cannot be expressed in a simple way, as the q GFs, in terms of the q SD $\Lambda_{qAB}(\omega)$ defined in terms of the same operators. This introduces in the formalism some intrinsic difficulties which, in view of the present still limited experience, can be overcome only under appropriate constraints such as, for instance, close to the extensive regime. Under the restrictive condition (56), the calculation of q CFs is sensibly simplified and we have shown how the formalism works in exploring the q-induced nonextensivity effects on the low-temperature properties of the model (78)–(80) for the high-density Bose gas with strong attraction between the particles [31].

In any case, it is desirable to test again the effectiveness of the q EM and q SD methods for other many-body problems in nonextensive quantum statistical mechanics. A preliminary study along this direction can be found in Ref. [46] where we have applied the q SDM to an isotropic Heisenberg model with long-range exchange interactions.

We wish to stress again that, in the context of practical calculations, the crucial problem remains to understand how one can calculate the relevant q CFs and the related q-thermodynamic quantities by using the general formulas (54) and (55) beyond the limiting condition (56).

In conclusion, in light of the great experience acquired in the extensive statistical mechanics [1–10], we believe that the q GF technique and, in particular, the q SDM may constitute a powerful tool of investigation also in nonextensive many-body theory.

APPENDIX A: TWO-TIME GREEN's FUNCTIONS AND SPECTRAL DENSITY METHOD IN NONEXTENSIVE CLASSICAL THERMOSTATISTICS

As mentioned in the Introduction, the pioneering framework of the two-time GF method in extensive classical sta-

tistical mechanics by Bogoliubov and Sadovnikov [8] has opened the concrete possibility to describe classical and quantum many-body systems on the same footing. Besides, in many physical situations (when the quantum effects are negligible), the use of the classical formalism may offer substantial advantages especially from the computational point of view because in the calculations one handles only functions and not operators.

Recently, the two-time GF technique and SDM have been formulated in nonextensive classical thermostatistics, within the OLM framework, in two of our papers [27,28] and conveniently applied to the ferromagnetic Heisenberg chain. For completeness, in this appendix, we review the basic equations to underline the main differences with the corresponding quantum case which has been the subject of the present article.

First, we note that the basic ingredients of the quantum Tsallis thermostatistics shortly reviewed in Sec. II remain formally valid for the classical framework. Here, $\rho=\rho(\mathbf{q},\mathbf{p})$ denotes the probability distribution defined in the space phase Γ, $\mathbf{q}=\{q_1,\ldots,q_{\mathcal{N}}\}$ and $\mathbf{p}=\{p_1,\ldots,p_{\mathcal{N}}\}$ are the generalized coordinates, and $\mathrm{Tr}(\cdots)$ stands for $\int(\cdots)d\Gamma=\int\Pi_{i=1}^{\mathcal{N}}dq_i dp_i$, for a classical system with Hamiltonian $H(\mathbf{q},\mathbf{p})$ and $\mathcal{N}$ degrees of freedom. As for the quantum case, also in this appendix we adopt the classical OLM representation [see Eqs. (10)–(13), (14a), (14b), and (15)–(17)].

1. Classical two-time q Green's functions and q-spectral density

In classical nonextensive thermostatistics, the two-time q GFs involving two classical observables are defined by [27]

$$G_{qAB}^{(\nu)}(t,t')=\langle\langle A(t);B(t')\rangle\rangle_q^{(\nu)}=\theta_\nu(t-t')\langle\{A(t),B(t')\}\rangle_q, \tag{A1}$$

where the set of Poisson brackets $\{\ldots,\ldots\}$ replaces the quantum commutator $-i[\ldots,\ldots]_\eta$ and the q expectation value is taken over the phase space within the OLM spirit. Here, $X(t)$ (with $X\equiv A,B$) represents a dynamical variable depending on time through the canonical coordinates with $X(t)=X(\mathbf{q}(t),\mathbf{p}(t))=X(e^{iL_Ht}\mathbf{q}(0),e^{iL_Ht}\mathbf{p}(0))=e^{iL_Ht}X(0)$, where $L_H=i\{H,\ldots\}$ is the *Liouville* operator. Hence, e^{iL_Ht} acts as a classical time-evolution operator which transforms the dynamical variable $X(0)\equiv X$ at initial time $t=0$ into the variable $X(t)$ at arbitrary time t, satisfying the Liouville equation $dX(t)/dt=iL_HX(t)$.

The main difference of the quantum and classical q GF methods lies in the relation between the q GFs and the q CFs. Actually, starting from the definitions of Poisson's brackets and q mean value, it can be shown that [27]

$$G_{qAB}^{(\nu)}(t,t')=q\beta\theta_\nu(t-t')\frac{d}{dt}F_{qA_qB_q}(t,t'), \tag{A2}$$

connecting $G_{qAB}^{(\nu)}(t,t')$ to the new generalized two-time q CF

$$F_{qA_qB_q}(t,t')=\langle A_q(t)B_q(t')\rangle_q \tag{A3}$$

of the two q dynamical variables A_q and B_q defined (working in the canonical ensemble) by

$$X_q=\frac{X}{\sqrt{C_q}},\quad C_q=1-\beta(1-q)(H-U_q)>0. \tag{A4}$$

Unfortunately, it is not possible to derive a direct relation between $G_{qAB}^{(\nu)}(t,t')$ and $F_{qAB}(t,t')$ as happens in the extensive classical case [9]. Nevertheless, using the series representation

$$\frac{1}{1-x}=1+2\sum_{n=1}^{\infty}J_n(nx), \tag{A5}$$

where $J_n(z)$ are the Bessel functions

$$J_n(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}z\right)^{2m+n}}{m!\,\Gamma(n+m+1)}, \tag{A6}$$

with $x=(1-q)\beta(H-U_q)$, we can formally write the expansion

$$G_{qAB}^{(\nu)}(t,t')=q\beta\theta_\nu(t-t')\frac{d}{dt}\Big[\langle A(t)B(t')\rangle_q+\sum_{n,m=1}^{\infty}C_q^{n,m}(\beta)\times\langle A(t)B(t')(H-U_q)^{2m+n}\rangle_q\Big], \tag{A7}$$

where

$$C_q^{m,n}(\beta)=\frac{2(-1)^m\left[\frac{n}{2}\beta(1-q)\right]^{2m+n}}{m!\,\Gamma(n+m+1)}. \tag{A8}$$

With the restriction $|x|<1$ the simplest series $(1-x)^{-1}=\Sigma_{n=0}^{\infty}x^n$ can be used to obtain the formula

$$G_{qAB}^{(\nu)}(t,t')=q\beta\theta_\nu(t-t')\frac{d}{dt}\Big[\langle A(t)B(t')\rangle_q+\sum_{n=1}^{\infty}\beta^n(1-q)^n\langle A(t)B(t')(H-U_q)^n\rangle_q\Big]. \tag{A9}$$

In particular, we have

$$G_{qAB}^{(\nu)}(t,t')\approx q\beta\theta_\nu(t-t')\frac{d}{dt}\langle A(t)B(t')\rangle_q, \tag{A10}$$

under the condition $|x|=|\beta(1-q)(H-U_q)|\ll 1$. This shows that only in the lowest order may the q GFs be directly related to $F_{qAB}(t,t')$ as in the extensive case [9]. Any way, we can improve systematically the approximation (A10), taking into account the successive terms in the expansion (A9) relating q GF to q CFs of increasing order.

Assuming time-translational invariance, all the spectral properties derived in Sec. III A remain formally unchanged, but now the classical q SD $\Lambda_{qAB}(\omega)$ is defined by

$$\Lambda_{qAB}(\omega)=i\int_{-\infty}^{+\infty}d\tau\,e^{i\omega\tau}\langle\{A(\tau),B\}\rangle_q. \tag{A11}$$

Besides, from Eqs. (A2) and (A3), one can show that [27]

$$\langle A_q(\tau)B_q\rangle_q=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\Lambda_{qAB}(\omega)e^{-i\omega\tau}}{q\beta\omega}. \tag{A12}$$

Then, using the series representation (A9), we get

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{qAB}(\omega)}{q\beta\omega} = \langle AB \rangle_q + \sum_{n,m=1}^{\infty} C_q^{n,m}(\beta) \langle AB(H-U_q)^{2m+n} \rangle_q, \tag{A13}$$

which corresponds to Eq. (54) and allows us to connect the classical q CFs with the related q SD. In the limit $q \to 1$, Eq. (A13) reproduces consistently the well-known exact extensive result [9]

$$\lim_{q\to 1} \langle AB \rangle_q = \langle AB \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\Lambda_{1AB}(\omega)}{\beta\omega}. \tag{A14}$$

In order to derive the spectral decomposition for $\Lambda_{qAB}(\omega)$, we now introduce a Hilbert space $\mathcal{S}_q$ of the classical dynamical variables with a scalar product defined conveniently by [9,58]

$$\langle A|B \rangle_q = \tilde{Z}_q \langle A^* B \rangle_q, \tag{A15}$$

with $\tilde{Z}_q = \int d\Gamma \{1-\beta(1-q)[H(\mathbf{q},\mathbf{p})-U_q]\}^{q/1-q}$. In this space one can consider the eigenvalue equation $L_H \Psi_k = \omega_k \Psi_k$ for the Hermitian Liouville operator. It is also immediate to prove that, if Ψ_k is an eigenfunction of L_H with (real) eigenvalues ω_k, then Ψ_k^* is also an eigenfunction of L_H with eigenvalue $-\omega_k$. If we assume that $\{\Psi_k\}$ is a complete set of orthonormal eigenfunctions, we can consider the expansions $A(\mathbf{q},\mathbf{p}) = \Sigma_k \langle \Psi_k^* | A \rangle_q \Psi_k^*(\mathbf{q},\mathbf{p})$ and $B(\mathbf{q},\mathbf{p}) = \Sigma_k \langle \Psi_k | B \rangle_q \Psi_k(\mathbf{q},\mathbf{p})$. Bearing this in mind, we can write [27]

$$\Lambda_{qAB}(\omega) = 2\pi q\beta\omega \tilde{Z}_q^{-1} \sum_k \langle \Psi_k | B_q \rangle_q \langle \Psi_k^* | A_q \rangle_q \delta(\omega - \omega_k), \tag{A16}$$

which is the desired classical q-spectral decomposition for $\Lambda_{qAB}(\omega)$. In particular, if $B=A^*$, we have that $\Lambda_{qAA^*}(\omega)$ is a real and positive-definite quantity. A consequence of Eq. (A16) is that $G_{qAB}(\omega)$ and $\langle A_q(t) B_q \rangle_q$ can be written as

$$G_{qAB}(\omega) = 2\pi q\beta \tilde{Z}_q^{-1} \sum_k \langle \Psi_k | B_q \rangle_q \langle \Psi_k^* | A_q \rangle_q \frac{\omega_k}{\omega - \omega_k} \tag{A17}$$

and

$$\langle A_q(\tau) B_q \rangle_q = \tilde{Z}_q^{-1} \sum_k \langle \Psi_k | B_q \rangle_q \langle \Psi_k^* | A_q \rangle_q e^{-i\omega_k \tau}. \tag{A18}$$

Thus, also in the classical case, the real poles of $G_{qAB}(\omega)$ represent the frequency spectrum of undamped oscillations.

2. Methods of calculation for classical two-time q Green's functions and q-spectral density

a. Classical equations-of-motion method

As in the quantum case, successive differentiations of Eq. (A1) with respect to $\tau = t - t'$ yield the infinite hierarchies of classical coupled EMs in τ and ω representations:

$$\frac{d}{d\tau} \langle\langle \mathcal{L}_H^m A(\tau); B \rangle\rangle_q^{(\nu)} = \delta(\tau) \langle \{\mathcal{L}_H^m A, B\} \rangle_q + \langle\langle \mathcal{L}_H^{m+1} A(\tau); B \rangle\rangle_q^{(\nu)}$$

$$(m = 0,1,2,\ldots) \tag{A19}$$

and

$$\omega \langle\langle \mathcal{L}_H^m A(\tau); B \rangle\rangle_{q,\omega}^{(\nu)} = i \langle \{\mathcal{L}_H^m A, B\} \rangle_q + i \langle\langle \mathcal{L}_H^{m+1} A(\tau); B \rangle\rangle_{q,\omega}^{(\nu)}$$

$$(m = 0,1,2,\ldots), \tag{A20}$$

respectively, where $\mathcal{L}_H = iL_H$ and $\mathcal{L}_H^m A$ means $\mathcal{L}_H^0 A = A$, $\mathcal{L}_H^1 A = \{A, H\}$, $\mathcal{L}_H^2 A = \{\{A, H\}, H\}$, and so on.

At this stage, the considerations made in Sec. IV B for solving the EMs in the quantum case apply for the classical one, too.

b. Classical q-spectral density method

From the definition $\Lambda_{qAB}(\tau) = i \langle \{A(\tau), B\} \rangle_q$ of the classical q SD in τ space, successive derivatives with respect to τ yield

$$\frac{d^m}{d\tau^m} \Lambda_{qAB}(\tau) = i \langle \{\mathcal{L}_H^m A(\tau), B\} \rangle_q \quad (m = 0,1,2,\ldots). \tag{A21}$$

Then, proceeding as in the quantum case, we easily find

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \omega^m \Lambda_{qAB}(\omega) = -(i)^{m-1} \langle \{\mathcal{L}_H^m A, B\} \rangle_q \quad (m = 0,1,2,\ldots). \tag{A22}$$

The quantity on the left-hand side will be called the m moment of $\Lambda_{qAB}(\omega)$, and relation (A22) can be seen as an infinite set of exact MEs or sum rules for the classical q SD.

As for the quantum counterpart, due to the possibility to evaluate the Poisson brackets and hence the q averages on the right-hand side of Eq. (A22), the m moments can be explicitly calculated without *a priori* knowledge of $\Lambda_{qAB}(\omega)$. So one can consider the sequence of equations, Eq. (A22), as a typical moment problem to determine the unknown function $\Lambda_{qAB}(\omega)$ and hence all the related macroscopic q quantities. At this stage, in view of the classical exact q-spectral decomposition (A16) for $\Lambda_{qAB}(\omega)$, the basic idea of the SDM [7] for classical q thermostatistics does not differ from the quantum case and all the considerations made in Sec. VI about the possible functional representations for the q SD preserve their validity in the context of the classical q many-body theory.

APPENDIX B: CALCULATION OF k SUMS AS $V \to \infty$ IN THE HIGH-DENSITY AND LOW-TEMPERATURE LIMITS

For the $(T=0)$-sums over $\mathbf{k}$ in Eqs. (111)–(113), with $V^{-1}\Sigma_{\mathbf{k}\neq 0}(\cdots) = (2\pi^2)^{-1} \int_0^\infty dk\, k^2 (\cdots)$ as $V \to \infty$ (all the summands depend only on $k = |\mathbf{k}|$) and the transformation $x = k(n\gamma)^{-1/4}$, we can write

$$\frac{1}{2V} \sum_{\mathbf{k}\neq 0} \frac{f_{\mathbf{k}} - \omega_{\mathbf{k}}}{\omega_{\mathbf{k}}} = \frac{(n\gamma)^{3/4}}{(2\pi)^2} \int_0^\infty dx \frac{\left[\frac{x^2}{2} - f(x)\right]^2}{f(x)}, \tag{B1}$$

$$\frac{1}{2V}\sum_{\mathbf{k}\neq 0}(f_{\mathbf{k}}-\omega_{\mathbf{k}})=\frac{(n\gamma)^{5/4}}{(2\pi)^2}\int_0^\infty dx\left[\frac{x^2}{2}-f(x)\right]^2, \quad \text{(B2)}$$

$$\frac{1}{2V}\sum_{\mathbf{k}\neq 0}k^2\frac{f_{\mathbf{k}}-\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}}=\frac{(n\gamma)^{5/4}}{(2\pi)^2}\int_0^\infty dx\frac{x^2\left[\frac{x^2}{2}-f(x)\right]^2}{f(x)}, \quad \text{(B3)}$$

where

$$f(x)=\left\{\frac{x^4}{4}+(n\gamma)^{1/2}x^2\left[\frac{(n\gamma)^{-1/2}}{(n\gamma)^{-1/2}+x^2}-\frac{e^{-(n\gamma)^{1/2}x^2R^2/4}}{\eta+1}\right]\right\}^{1/2}. \quad \text{(B4)}$$

Following Babichenko [31], with $(n\gamma)^{1/2}\gg 1$ one can now neglect the term $(n\gamma)^{-1/2}$ with respect to x^2 and the Gaussian part in Eq. (B4). This is quite legitimate since the basic contribution to the previous integrals is made by $x\sim 1$.

Then, one finds

$$\frac{1}{2V}\sum_{\mathbf{k}\neq 0}\frac{f_{\mathbf{k}}-\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}}\simeq a(n\gamma)^{3/4}, \quad \text{(B5)}$$

$$\frac{1}{2V}\sum_{\mathbf{k}\neq 0}(f_{\mathbf{k}}-\omega_{\mathbf{k}})\simeq\frac{4}{5}b(n\gamma)^{5/4}, \quad \text{(B6)}$$

$$\frac{1}{2V}\sum_{\mathbf{k}\neq 0}k^2\frac{f_{\mathbf{k}}-\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}}\simeq\frac{2}{5}b(n\gamma)^{5/4}, \quad \text{(B7)}$$

where

$$a=\frac{1}{(2\pi)^2}\int_0^\infty dx\frac{\left[\left(1+\frac{x^4}{4}\right)^{1/2}-\frac{x^2}{2}\right]^2}{\left(1+\frac{x^4}{4}\right)^{1/2}}=\frac{\sqrt{2}}{48\pi^{5/2}}\Gamma^2\left(\frac{1}{4}\right), \quad \text{(B8)}$$

$$b=\frac{1}{(2\pi)^2}\int_0^\infty dx\frac{\left(1+\frac{x^4}{4}\right)^{1/2}-\frac{x^2}{2}}{\left(1+\frac{x^4}{4}\right)^{1/2}}=\frac{\sqrt{2}}{4\pi^{5/2}}\Gamma^2\left(\frac{3}{4}\right). \quad \text{(B9)}$$

Concerning the T-dependent sums over $\mathbf{k}$ in Eqs. (111)–(113), in the low-temperature limit only small values of k contribute to the integrals which contain exponential functions. So, with the additional condition $(n\gamma)^{1/2}\gg 1$, all the integrals can be exactly computed and we get, to leading order in T,

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_{\mathbf{k}}}{e^{\beta\omega_{\mathbf{k}}}-1}\simeq\frac{\pi^2}{30}\left(\frac{\eta+1}{\eta}\right)^{3/2}(n\gamma)^{-3/2}T^4, \quad \text{(B10)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{f_{\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{1}{e^{\beta\omega_{\mathbf{k}}}-1}\simeq\frac{1}{12}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^2, \quad \text{(B11)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_{\mathbf{k}}^2e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\simeq\frac{2\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{3/2}(n\gamma)^{-3/2}T^5, \quad \text{(B12)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{f_{\mathbf{k}}e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\simeq\frac{1}{6}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^3, \quad \text{(B13)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{\omega_{\mathbf{k}}^2\varphi(k)e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\simeq\frac{2\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{1/2}\gamma(n\gamma)^{-3/2}T^5, \quad \text{(B14)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{k^2\varphi(k)e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\simeq\frac{2\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{3/2}\gamma(n\gamma)^{-5/2}T^5, \quad \text{(B15)}$$

$$\frac{1}{V}\sum_{\mathbf{k}\neq 0}\frac{f_{\mathbf{k}}\omega_{\mathbf{k}}^2e^{\beta\omega_{\mathbf{k}}}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}\simeq\frac{2\pi^2}{15}\left(\frac{\eta+1}{\eta}\right)^{1/2}(n\gamma)^{-1/2}T^5. \quad \text{(B16)}$$

Inserting the previous results into Eqs. (111)–(113), one immediately obtains expressions (114)–(116) for the depletion of the condensate, the chemical potential, the q internal energy density, and hence the q specific heat (117) as a function of T and n in the high-density and low-temperature limits.

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